MPM2D: Principles of Mathematics

Quadratic Relations

Quadratics In Vertex Form

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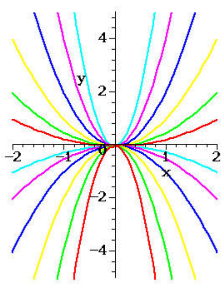
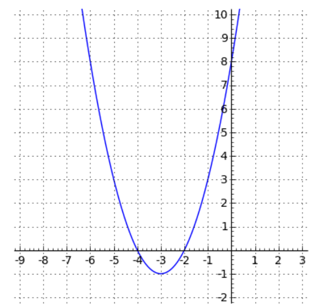


Quadratic Relations

Recap

State the key properties of the graph of $y = x^2 + 6x + 8$.



Quadratic Relations

The key features of the graph of $y = x^2 + 6x + 8$ are:

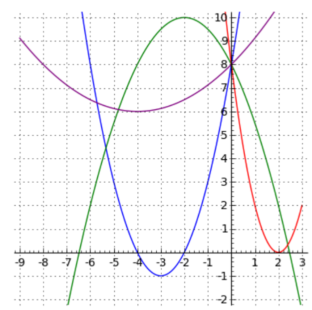
- the parabola opens upward
- the y-intercept is at $(0, 8)$
- there are x-intercepts at $(-4, 0)$ and $(-2, 0)$
- the vertex is a minimum, located at $(-3, -1)$
- the axis of symmetry has equation $x = -3$

Note that only the direction of opening and the y-intercept are easily identified in the equation.

The x-intercepts and the vertex are harder to identify without resorting to formulae.

Quadratic Relations

Since there is an infinity of parabolas with the same y-intercept, knowing only its location is not very useful.



Quadratic Relations

In the earlier transformations investigation, we noted the following two facts:

- The vertex of $y = x^2 + k$ is at $(0, k)$.
- The vertex of $y = (x - h)^2$ is at $(h, 0)$.

Recall that the sign of h appears "opposite" its actual coordinate, due to the negative sign inside of the brackets.

Vertical and horizontal transformations are independent, and have no effect on the other.

This means that a quadratic relation of the form $y = (x - h)^2 + k$ will have its vertex at (h, k).

Quadratic Relations

Example

Determine the coordinates of the vertex of $y = (x - 5)^2 + 3$.

In the equation, $h = 5$ and $k = 3$, so the vertex is at $(5, 3)$.

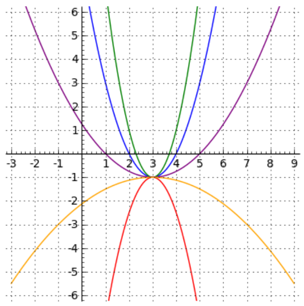
Example

Determine the coordinates of the vertex of $y = (x + 7)^2 + 1$.

In the equation, $h = -7$ $(x - (-7) = x + 7)$ and $k = 1$, so the vertex is at $(-7, 1)$.

Quadratic Relations

While this is useful, there is still an infinity of parabolas that have a common vertex.



Quadratic Relations

Recall from the investigation that the parabola described by $y = ax^2$ opens upward if $a > 0$, and downward if $a < 0$.

This narrows down the field of possibilities for a parabola with a given vertex, but is not sufficient to identify a specific parabola.

However, the value of a also indicates whether the parabola has been vertically stretched (made taller) or compressed (made smaller).

It is this fact that gives the remaining information to accurately graph a parabola.

Quadratic Relations

Consider the quadratic relation $y = x^2$, and its finite differences.

x	y		
0	0	$\Delta 1$	
1	1	1	$\Delta 2$
2	4	3	2
3	9	5	2
4	16	7	2

When $a = 1$, the value of the second differences is 2.

The pattern in the first differences begins $1, 3, 5, \ldots$.

Quadratic Relations

Now consider the quadratic relation $y = 2x^2$, and its finite differences.

x	y		
0	0	$\Delta 1$	
1	2	2	$\Delta 2$
2	8	6	4
3	18	10	4
4	32	14	4

When $a = 2$, the value of the second differences is 4.

The pattern in the first differences begins $2, 6, 10, \ldots$.

Quadratic Relations

What will be the constant value of the second differences for $y = 3x^2$, and the pattern in the first differences?

x	y		
0	0	$\Delta 1$	
1	3	3	$\Delta 2$
2	12	9	6
3	27	15	6
4	48	21	6

When $a = 3$, the value of the second differences is 6.

The pattern in the first differences begins $3, 9, 15, \ldots$.

Quadratic Relations

For any quadratic relation, the constant value of the second differences is twice the value of its leading coefficient, a.

The pattern in the first differences is often referred to as the "step pattern", because it resembles a flight of steps of increasing height when graphed.

Vertex Form of a Quadratic

A quadratic relation in *vertex form*, $y = a(x - h)^2 + k$, has its vertex at (h, k). It opens upward if $a > 0$, and downward if $a < 0$. The "step pattern" from the vertex is given by $a, 3a, 5a, \ldots$.

Vertex form is extremely useful, because it is possible to graph a parabola using the step pattern, beginning at the vertex.

Quadratic Relations

Example

Graph the parabola given by $y = (x-2)^2 - 3$.

The vertex of the parabola is at $(2, -3)$.

Since $a = 1$, the step pattern is $1, 3, 5, \ldots$.

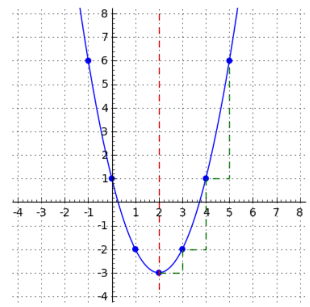
From $(2, -3)$, move 1 unit right and 1 unit up, to $(3, -2)$.

From $(3, -2)$, move 1 unit right and 3 units up, to $(4, 1)$.

From $(4, 1)$, move 1 unit right and 5 units up, to $(5, 6)$.

Use symmetry to copy these three points in the axis of symmetry, $x = 2$.

Quadratic Relations



Quadratic Relations

Example

Graph the parabola given by $y = -2(x+4)^2 + 7$.

The vertex of the parabola is at $(-4, 7)$.

Since $a = -2$, the step pattern is $-2, -6, -10, \ldots$.

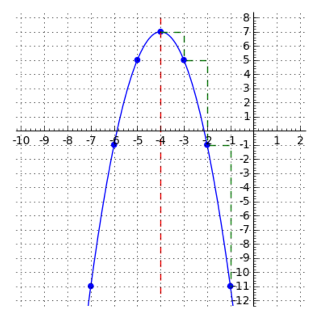
From $(-4, 7)$, move 1 unit right and 2 units down, to $(-3, 5)$.

From $(-3, 5)$, move 1 unit right and 6 units down, to $(-2, -1)$.

From $(-2, -1)$, move 1 unit right and 10 units down, to $(-1, -11)$.

Use symmetry to copy these three points in the axis of symmetry, $x = -4$.

Quadratic Relations



Quadratic Relations

Example

Determine an equation for a quadratic relation with its vertex at $V(-3, 12)$ if it passes through the point $P(-1, 4)$.

Substitute $h = -3$, $k = 12$, $x = -1$ and $y = 4$ into the vertex form of a quadratic relation, and solve for a.

$$\begin{aligned} y &= a(x-h)^2 + k \\ 4 &= a(-1+3)^2 + 12 \\ 4 &= a(2)^2 + 12 \\ -8 &= 4a \\ a &= -2 \end{aligned}$$

An equation that meets these criteria is $y = -2(x+3)^2 + 12$.

Questions?