MPM2D: Principles of Mathematics

Tangents

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Equations of Circles

Recap

Determine the equation and length of the radius of a circle, centred at the origin, that passes through the point $P(9, -3)$.

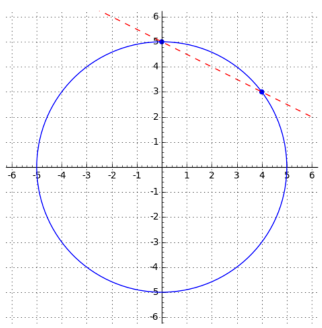
Substitute $x = 9$ and $y = -3$ into the equation of a circle.

$$\begin{aligned} 9^2 + (-3)^2 &= 81 + 9 \\ &= 90 \end{aligned}$$

The circle has equation $x^2 + y^2 = 90$, and radius $r = \sqrt{90} = 3\sqrt{10}$.

Secants and Chords

Consider the circle and line shown below.



Secants and Chords

The line intersects the circle at two points, $P(0, 5)$ and $Q(4, 3)$.

A straight line that connects any two points on a graph is called a *secant*.

A line segment that connects two points on a circle is called a *chord*. In the diagram, the secant through P and Q contains the chord PQ.

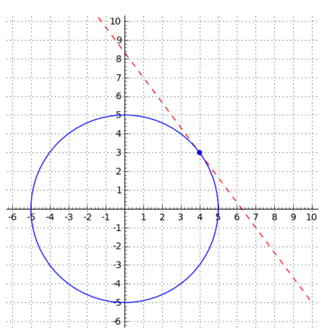
The equation of a secant can be found by using the two points on the circle to calculate the slope, then substituting into the equation of a straight line.

$$m_S = \frac{3 - 5}{4 - 0} = -\tfrac{1}{2}$$

$$y = -\tfrac{1}{2}x + 5$$

Tangents

Now consider the circle and line shown below.



Tangents

The line intersects the circle at one point, $P(4, 3)$.

A straight line that "just touches" a point on a graph is called a *tangent*.

In this case, P is the *point of tangency* to the line.

When tracing along the graph, a tangent to point P provides the best straight-line approximation to the graph at P.

How can we determine the equation of the tangent?

Tangents

If O is the origin, and the circle is centred at O, then a tangent to a circle at point P has a slope that is perpendicular to the radius OP.

In this case, the slope of OP is $m_{OP} = \frac{3}{4}$, so the slope of the tangent must be $m_T = -\frac{4}{3}$.

Since we know the coordinates of P, we can substitute the values of x and y to determine the equation of the tangent.

$$\begin{aligned} 3 &= -\tfrac{4}{3}(4) + b \\ 3 &= -\tfrac{16}{3} + b \\ 9 &= -16 + 3b \\ b &= \tfrac{25}{3} \end{aligned}$$

The equation of the tangent is $y = -\frac{4}{3}x + \frac{25}{3}$.

Tangents

This process leads us to an equation for a tangent to a circle, centred at the origin.

Equation of a Tangent to a Circle Centred at the Origin

The equation of a tangent to a circle, centred at the origin, passing through $P(x_P, y_P)$, is $y = -\frac{x_P}{y_P}x + \frac{{x_P}^2 + {y_P}^2}{y_P}$.

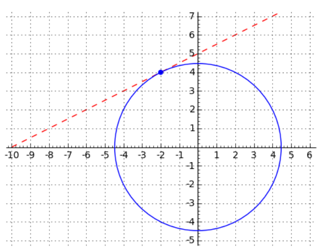
Using the previous example, the equation of the tangent at $(4, 3)$ is $y = -\frac{4}{3}x + \frac{4^2+3^2}{3} = -\frac{4}{3}x + \frac{25}{3}$.

Knowing the formula can be a useful shortcut, but it is fairly complex, whereas the process for determining the equation for a tangent may be more intuitive.

Tangents

Example

Determine the equation of the tangent to a circle, centred at the origin, that passes through $P(-2, 4)$.



Tangents

Determine the slope from the origin to P.

$$m_{OP} = -\frac{4}{2} = -2$$

Therefore, the slope of the tangent is $m_T = \frac{1}{2}$. Use $x = -2$ and $y = 4$ in $y = mx + b$.

$$\begin{aligned} 4 &= \tfrac{1}{2}(-2) + b \\ 4 &= -1 + b \\ b &= 5 \end{aligned}$$

The equation of the tangent is $y = \frac{1}{2}x + 5$.

This can also be determined using the formula, $y = -\frac{(-2)}{4}x + \frac{(-2)^2+4^2}{4} = \frac{1}{2}x + 5$.

Tangents

Example

The line $y = 3x - 10$ is tangent a circle, centred at the origin, when $x = 3$. Determine the equation and radius of the circle.

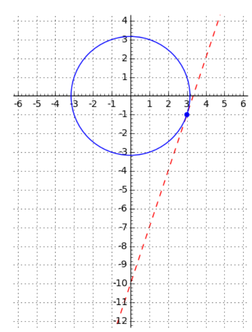
When $x = 3$, $y = 3(3) - 10 = -1$. Therefore, the point of tangency is $(3, -1)$.

Determine the equation of the circle passing through $(3, -1)$.

$$\begin{aligned} 3^2 + (-1)^2 &= 9 + 1 \\ &= 10 \end{aligned}$$

Therefore, the equation of the circle is $x^2 + y^2 = 10$, and its radius is $\sqrt{10} \approx 3.2$.

Tangents



Tangents

Example

Determine the equation of the tangent to the circle, centred at the origin with radius 13 units, when $x = 5$.

Since $r = 13$, $r^2 = 169$, so the equation of the circle is $x^2 + y^2 = 169$.

Substitute $x = 5$ into the circle's equation.

$$\begin{aligned} 5^2 + y^2 &= 169 \\ y^2 &= 144 \\ y &= \pm 12 \end{aligned}$$

There are two possible points of tangency when $x = 5$: $P(5, 12)$ and $Q(5, -12)$.

Tangents

For the tangent at $P(5, 12)$, the slope of OP is $m_{OP} = \frac{12}{5}$.

Therefore, the tangent has a slope of $m_T = -\frac{5}{12}$.

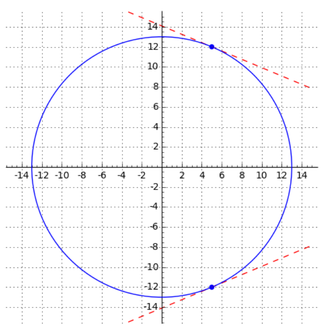
Use the slope of the tangent, and point P, to determine its equation.

$$\begin{aligned} 12 &= -\tfrac{5}{12}(5) + b \\ 12 &= -\tfrac{25}{12} + b \\ 144 &= -25 + 12b \\ b &= \tfrac{169}{12} \end{aligned}$$

The equation of the tangent to $P(5, 12)$ is $y = -\frac{5}{12}x + \frac{169}{12}$.

Similarly, the tangent to $Q(5, -12)$ is $y = \frac{5}{12}x - \frac{169}{12}$.

Tangents



Questions?