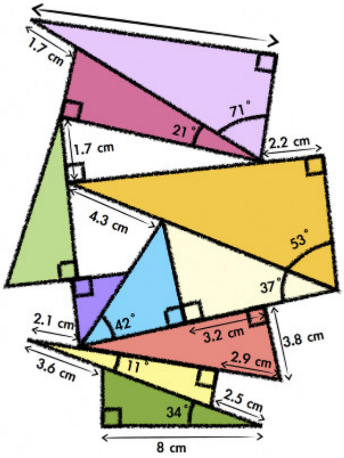
MPM2D: Principles of Mathematics

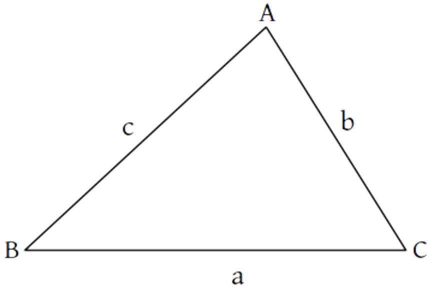
The Cosine Law

J. Garvin



Cosine Law

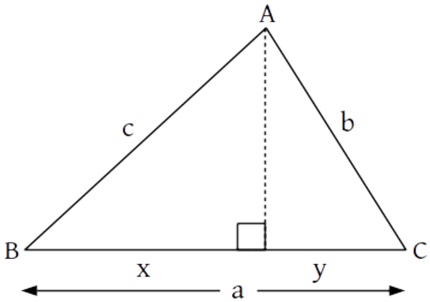
Consider the *oblique* triangle shown below, where $\angle A$, side b and side c are all known values.



How can we determine the length of a?

Cosine Law

We can construct two right triangles, as shown, dividing side b into two sections with lengths x and y.



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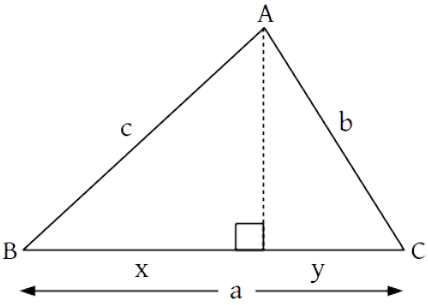

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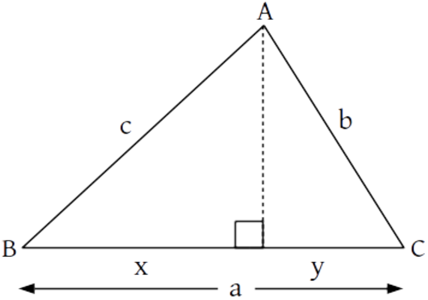

In the left triangle, $\cos B = \frac{x}{c}$, so $x = c \cdot \cos B$.

In the right triangle, $\cos C = \frac{y}{b}$, so $y = b \cdot \cos C$.

Cosine Law

Since $a = x + y$, $a = c \cdot \cos B + b \cdot \cos C$.

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Multiplying both sides of this equation by a produces the following relationship:

$$a^2 = a \cdot c \cdot \cos B + a \cdot b \cdot \cos C$$

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By dividing sides b and c into right triangles using the same method, we obtain two other relationships:

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$$c^2 = a \cdot c \cdot \cos B + b \cdot c \cdot \cos A$$

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$$c^2 = a \cdot c \cdot \cos B + b \cdot c \cdot \cos A$$

These can be rearranged to isolate the coloured terms:

$$a \cdot b \cdot \cos C = b^2 - b \cdot c \cdot \cos A$$

$$a \cdot c \cdot \cos B = c^2 - b \cdot c \cdot \cos A$$

Cosine Law

Now we can substitute the coloured terms into the first equation.

$$a^2 = a \cdot c \cdot \cos B + a \cdot b \cdot \cos C$$
$$a^2 = (c^2 - b \cdot c \cdot \cos A) + (b^2 - b \cdot c \cdot \cos A)$$
$$a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos A$$

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This is known as the Law of Cosines, or Cosine Law.

Law of Cosines

Given ΔABC, $a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos A$.

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Law of Cosines

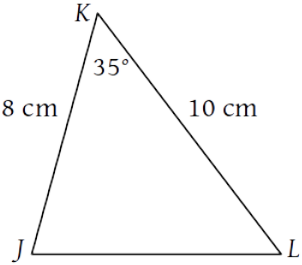
Given ΔABC, $a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos A$.

Note that the Cosine Law uses an angle that falls between two adjacent sides.

Cosine Law

Example

Determine $|JL|$.



Cosine Law

$$|JL|^2 = |JK|^2 + |KL|^2 - 2 \cdot |JK| \cdot |KL| \cdot \cos K$$

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Cosine Law

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\begin{aligned}
|JL|^2 &= |JK|^2 + |KL|^2 - 2 \cdot |JK| \cdot |KL| \cdot \cos K \\
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|JL| &\approx \sqrt{32.935673}
\end{aligned}
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|JL| &\approx \sqrt{32.935673} \\
|JL| &\approx 5.74 \text{ cm}
\end{aligned}
$$

Cosine Law

Like the Sine Law, it is possible to use the Cosine Law to find the measure of an angle.

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Thus, given the three side lengths of a triangle, it is possible to find the measure of the angle that is opposite the side that is isolated in the formula.

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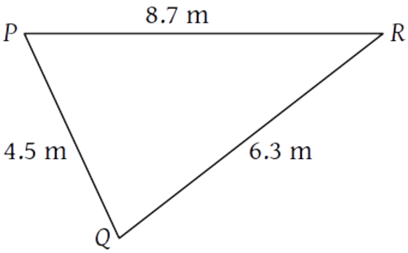
Instead, we can use algebra to isolate the variable representing the angle.

Cosine Law

Example

Determine the measure of $\angle R$.



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$$|PQ|^2 = |PR|^2 + |QR|^2 - 2 \cdot |PR| \cdot |QR| \cdot \cos R$$

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-95.13 &= -109.62\cos R
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-95.13 &= -109.62\cos R \\
\cos R &= \tfrac{95.13}{109.62} \\
R &= \cos^{-1}\left(\tfrac{95.13}{109.62}\right) \\
R &\approx 29.8^\circ
\end{aligned}
$$

Questions?