MPM2D: Principles of Mathematics

Applications of Quadratic Relations

Part 2: Min/Max Problems

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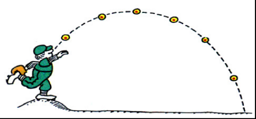


Maximum of a Quadratic Relation

Recall

Determine the maximum value of $y = -2x^2 + 28x - 100$.

$$\begin{aligned} y &= -2(x^2 - 14x) - 100 \\ y &= -2(x^2 - 14x + 49 - 49) - 100 \\ y &= -2([x-7]^2 - 49) - 100 \\ y &= -2(x-7)^2 - 2 \end{aligned}$$

The maximum value is -2, when $x = 7$.

Applications of Quadratic Relations

Many applications of quadratic relations involve finding the minimum or maximum value.

For example, the maximum height of a toy rocket can be calculated by modelling its flight path with a quadratic equation and determining the location of the vertex.

These problems are often referred to as "*min/max*" problems.

Most of the time, words such as "greatest", "least", "biggest", "smallest", "optimal", etc. indicate min/max problems.

To determine the location of the vertex, either complete the square or use partial factoring.

Applications of Quadratic Relations

Example

A firework, launched into the air with a velocity of 58.8 m/s from a height of 2 m, explodes at its highest point. Its height, h metres, is given by $h = -4.9t^2 + 58.8t + 2$, where t is the time in seconds. When does the firework explode? How high is it?

The highest point will be the vertex of its parabolic path.

$$\begin{aligned} h &= -4.9(t^2 - 12t) + 2 \\ h &= -4.9(t^2 - 12t + 36 - 36) + 2 \\ h &= -4.9(t-6)^2 + 178.4 \end{aligned}$$

The vertex is at $(6, 178.4)$. Therefore, the maximum height of 178.4 m occurs at 6 sec.

Applications of Quadratic Relations

Example

Determine the values of two numbers, one 10 greater than another, if the sum of their squares is a minimum. What is the minimum?

Let the first number be x. Then the other number is $x + 10$.

An equation representing the sum of their squares is $y = x^2 + (x+10)^2$.

Expand and simplify this equation.

$$\begin{aligned} y &= x^2 + (x^2 + 20x + 100) \\ y &= 2x^2 + 20x + 100 \end{aligned}$$

Applications of Quadratic Relations

Complete the square to find the minimum value.

$$\begin{aligned} y &= 2(x^2 + 10x) + 100 \\ y &= 2(x^2 + 10x + 25 - 25) + 100 \\ y &= 2(x+5)^2 + 50 \end{aligned}$$

The minimum value of 50 occurs when $x = -5$.

Therefore, the two numbers are -5 and 5.

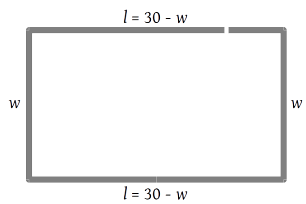
Note that $(-5)^2 + 5^2 = 25 + 25 = 50$.

Applications of Quadratic Relations

Example

A 60 cm wire is bent into a rectangle. Determine the maximum area and the dimensions that produce it.

Let w be the width of the rectangle. Then the length is $l = \frac{60-2w}{2} = 30 - w$.



Applications of Quadratic Relations

The area of the rectangle, then, is $A = w(30 - w)$.

Since the expression is in factored form, use the x-intercepts to determine the location of the vertex.

$$0 = w(30 - w)$$
$$w = 0 \text{ or } 30$$

The x-coordinate of the vertex will be at $\frac{0+30}{2} = 15$.

When $x = 15$, the area is $A = 15(30 - 15) = 225$ cm^2.

The dimensions are 15×15 cm. The maximum area results when the rectangle is a square.

Applications of Quadratic Relations

Example

A t-shirt manufacturer typically sells 500 shirts per week to distributors for \$4.00 each. For each \$0.50 reduction in price, she estimates she can sell an additional 20 shirts. How much should she charge per shirt to maximize her revenue?

Revenue is defined as

$$\text{revenue} = \text{unit price} \times \text{number of units sold}$$

For example, selling 500 shirts for \$4.00 each results a revenue of $500 \times 4 = \$2\,000.00$.

A lower price may result in more sales, but at a lower income. A balance between price and number of sales is needed.

Applications of Quadratic Relations

Let n be the number of price reductions.

The number of shirts sold per week will be $500 + 20n$, and the unit price of a shirt will be $4 - 0.50n$.

A revenue equation is $R = (500 + 20n)(4 - 0.50n)$.

$$0 = (500 + 20n)(4 - 0.50n)$$
$$n = -25 \text{ or } 8$$

The maximum revenue will result when there are $\frac{-25+8}{2} = -8.5$ reductions in price.

This means that the manufacturer should *increase* the price of a shirt by $8.5 \times \$0.50 = \$4.25$ to maximize her profit.

Therefore, the unit price should $\$4.00 + \$4.25 = \$8.25$, resulting in a sale of $500 + 20(-8.5) = 330$ shirts.

The maximum revenue is $330 \times \$8.25 = \$2\,722.50$.

Questions?