Solving Linear Systems Graphically

J. Garvin

Equations of Lines

Recap

Determine whether the lines \( y = \frac{5}{3}x + 4 \) and \( 5x - 3y = 24 \) are coincident, parallel and distinct, or neither.

Rearrange the second equation into slope-intercept form to compare the slopes.

\[
\begin{align*}
5x - 3y &= 24 \\
-3y &= -5x + 24 \\
y &= \frac{5}{3}x - 8 
\end{align*}
\]

Since the slopes are the same, the lines are either coincident or parallel and distinct.

Since the \( y \)-intercepts are different, the two lines must be parallel and distinct.

Linear Systems

Two or more linear equations that are considered simultaneously is called a linear system.

These may be expressed in either slope-intercept (\( y = mx + b \)) or standard (\( Ax + By = C \)) form.

The solution to a linear system is a set of values that satisfies all equations.

For example, the linear system with equations \( y = 5x - 12 \) and \( y = 3x - 4 \) has the solution \( x = 4, \ y = 8 \), since these values make both equations true.

\[
\begin{align*}
8 &= 5(4) - 12 \\
8 &= 3(4) - 4 
\end{align*}
\]

A graph of the two relations is below, showing the point of intersection at (4, 8).

Linear Systems

When graphed, the linear relations \( y = 5x - 12 \) and \( y = 3x - 4 \) make two straight lines.

Since these lines have different slopes, they must intersect at a unique point.

Since \( x = 4, \ y = 8 \) is a solution to the linear system, this corresponds to the point (4, 8) on the graph.

When represented graphically, the solution to a linear system is the point of intersection (POI) of the lines.

Linear Systems

Sometimes, two lines are parallel to each other.

Since two parallel lines will never intersect, there will not be a point of intersection.

This implies that a linear system represented by two parallel lines will have no solution.

In other cases, the same line may be described in multiple ways (i.e. in slope-intercept form and in standard form).

If both lines are coincident, there will be an infinite number of solutions, since any values of \( x \) and \( y \) that satisfy the first equation will also satisfy the second.
Linear Systems

Number of Solutions to a Linear System

For a linear system, the number of solutions will be:
- one, if the lines are not parallel,
- zero, if the lines are parallel and distinct, or
- infinite, if the lines are coincident.

Each scenario is depicted below.

Example

Solve the linear system with equations $y = 2x - 5$ and $y = -\frac{1}{2}x + 2$.

Since the two lines have different slopes, they must intersect at a single point.

The first line has a $y$-intercept at $-5$. Beginning at this point, move up 2 units and right 1 unit.

The second line has a $y$-intercept at 2. Beginning at this point, move down 1 unit and right 3 units.

A graph of the lines is below, showing a point of intersection at $(3, 1)$. The solution is $x = 3, y = 1$.

Example

Solve the linear system with equations $y = 3x + 5$ and $y = 2x + 3$.

Even though both slopes are positive, they are different, so the lines will intersect at a single point.

The first line has a $y$-intercept at 5. Beginning at this point, move up 3 units and right 1 unit.

The second line has a $y$-intercept at 3. Beginning at this point, move up 2 units and right 1 unit.

Example

Solve the linear system with equations $y = \frac{3}{2}x - 4$ and $3x - 2y = -2$.

Since the second line is expressed using standard form, and since 3 is not a factor of $-2$, convert it to slope-intercept form.

$$3x - 2y = -2$$
$$-2y = -3x - 2$$
$$y = \frac{3}{2}x + 1$$

Since the slopes of the lines are identical, but the $y$-intercepts are different, the lines must be parallel and distinct. Therefore, there is no solution.
Linear Systems
A graph of the lines is below, confirming that the two lines are parallel.

Example
Solve the linear system with equations $y = -\frac{1}{2}x + 2$ and $x + 2y = 4$.

Again, convert the second line to slope-intercept form.

\[
\begin{align*}
    x + 2y &= 4 \\
    2y &= -x + 4 \\
    y &= -\frac{1}{2}x + 2
\end{align*}
\]

Since the slopes of the lines are identical, and the $y$-intercepts are the same, the lines must be coincident. Therefore, there are an infinite number of solutions.

Questions?