MHF4U: Advanced Functions

Transformations of Polynomial Functions

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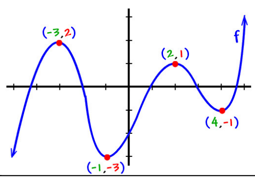


Transformations

In most cases, the graph of a function is similar to a simpler version, but may appear stretched, shifted or reflected to some extent.

The simplest version of a function that possesses all of the same characteristics of the derived function is called a *parent function* or a *base function*.

If we know information about a particular base function, it may be possible to sketch a graph of the derived function by analyzing the transformations that have been applied to the base function.

Transformations

Transformations of Polynomial Functions

A polynomial of the form $f(x) = a(b(x-c))^n + d$, where a, b, c and d are real constants, and n is a natural number, is a transformation of some power function $g(x) = x^n$.

In the form above:

- a is a vertical stretch/compression, and possibly a reflection
- b is a horizontal stretch/compression, and possibly a reflection
- c is a horizontal translation
- d is a vertical translation

Vertical and Horizontal Transformations

Transformations may be applied vertically or horizontally.

In the function $f(x) = a(b(x-c))^n + d$, both a and d are vertical transformations. They appear "outside" of the function in its equation.

The parameters b and c are horizontal transformations. They appear "inside" of the function's equation.

Horizontal transformations may appear to behave opposite to intuition: larger numbers for b compress the graph, smaller numbers stretch it, and the parameter c seems to shift the graph in the opposite direction.

Vertical Stretches/Compressions

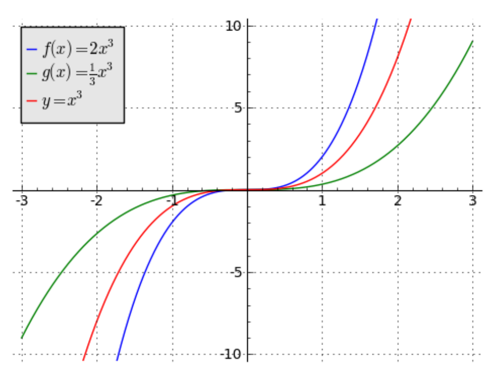
Example

Sketch graphs of $f(x) = 2x^3$ and $g(x) = \frac{1}{3}x^3$.

For $f(x)$, $|a| > 1$, so it has a vertical stretch by a factor of 2. All points are twice as far from the x-axis as they are on the graph of $y = x^3$.

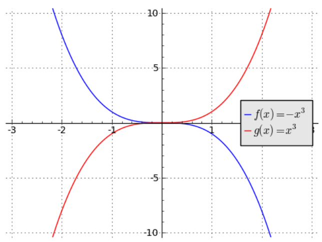
For $g(x)$, $0 < |a| < 1$, so it has a vertical compression by a factor of $\frac{1}{3}$. All points are one-third as far from the x-axis as they are on the graph of $y = x^3$.

Vertical Stretches/Compressions



Vertical Reflections

If $a < 0$, then a transformed power function has undergone a vertical reflection (reflection in the x-axis).



Horizontal Stretches/Compressions

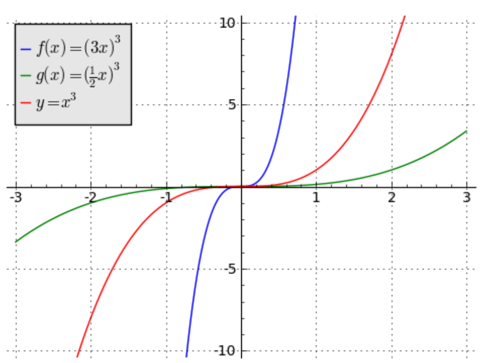
Example

Sketch graphs of $f(x) = (3x)^3$ and $g(x) = \left(\frac{1}{2}x\right)^3$.

For $f(x)$, $|b| > 1$, so it has a horizontal *compression* by a factor of $\frac{1}{3}$. All points are three times as far from the $f(x)$-axis as they are on the graph of $y = x^3$.

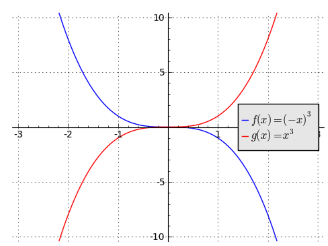
For $g(x)$, $0 < |b| < 1$, so it has a horizontal *stretch* by a factor of 2. All points are twice as far from the $f(x)$-axis as they are on the graph of $y = x^3$.

Horizontal Stretches/Compressions



Horizontal Reflections

If $b < 0$, then a transformed power function has undergone a horizontal reflection (reflection in the $f(x)$-axis).



Vertical and Horizontal Translations

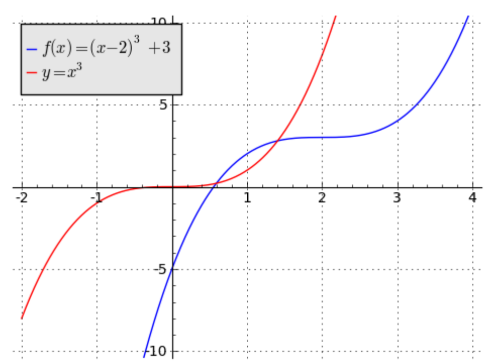
Example

Sketch a graph of $f(x) = (x - 2)^3 + 3$.

The graph of $f(x)$ has two transformations: a horizontal translation 2 units to the *right*, and a vertical translation 3 units up.

Neither transformation affects the shape of the graph, only its position.

Vertical and Horizontal Translations



Identifying Transformations From an Equation

Example

Identify the base function, and the transformations applied to it, to create the function $f(x) = 2(3x - 1)^3 - 5$.

The base function is $y = x^3$.

The 2 indicates a vertical stretch by a factor of 2.

The 3 indicates a horizontal compression by a factor of $\frac{1}{3}$.

There is a horizontal translation $\frac{1}{3}$ of a unit to the right, since the equation can be written $f(x) = 2\left(3\left(x - \frac{1}{3}\right)\right)^3 - 5$.

Finally, there is a vertical translation down 5 units.

Graphing Transformed Functions

Example

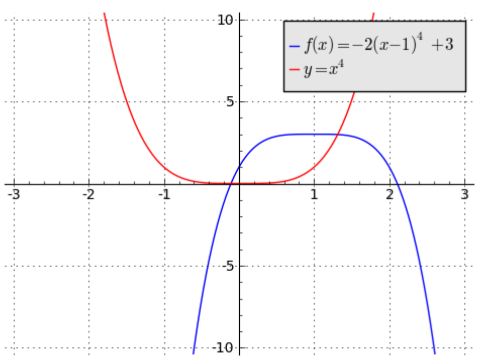
Sketch a graph of $f(x) = -2(x - 1)^4 + 3$.

The base power function, $y = x^4$, has Q2-Q1 end behaviour and its "vertex" at the origin.

$f(x)$ has a vertical reflection, so its end behaviour is Q3-Q4.

There is a vertical stretch by a factor of 2, a horizontal translation 1 unit to the right, and a vertical translation 3 units up.

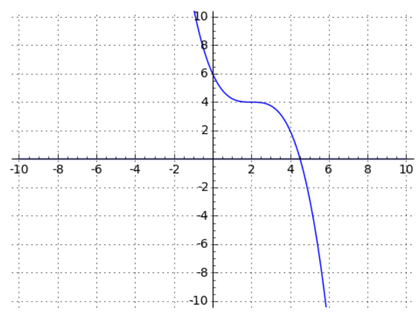
Graphing Transformed Functions



Determining Equations From Graphs

Example

Determine an equation for the function shown below.



Determining Equations From Graphs

The function has Q2-Q4 end behaviour, so it has an odd degree (likely cubic) and negative leading coefficient.

The "pivot point" of the function is at $(2, 4)$, indicating a vertical translation up 4 units and a horizontal translation right 2 units.

To determine if a vertical stretch has occurred, note that the function has an $f(x)$-intercept at 6.

To go from $(2, 4)$ to $(0, 6)$, there is a vertical change of 2 for a horizontal change of 2.

For the parent function $y = x^3$, there is a horizontal change of 2 from $(0, 0)$ to $(2, 8)$, resulting in a vertical change of 8.

Thus, there is a vertical compression by a factor of $\frac{1}{4}$.

A possible equation, then, is $f(x) = -\frac{1}{4}(x - 2)^3 + 4$.

Questions?