MHF4U: Advanced Functions

Solving Trigonometric Equations

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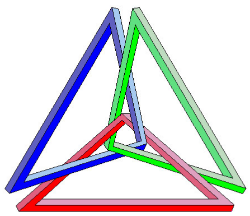

Solving Trigonometric Equations

Like other types of equations (linear, quadratic, etc.), we can solve trigonometric equations by isolating the independent variable.

In some cases, this can be done using simple algebra, whereas in other cases, it may be necessary to factor a trigonometric expression first.

When possible, use exact values to preserve accuracy.

Solving Trigonometric Equations

Example

Solve $2\sin x - 1 = 0$ on the interval $[0, 2\pi]$.

$$\begin{aligned} 2\sin x &= 1 \\ \sin x &= \tfrac{1}{2} \\ x &= \sin^{-1}\left(\tfrac{1}{2}\right) \end{aligned}$$

Since sine is positive in quadrants 1 and 2, there are two solutions.

In quadrant 1, $x = \frac{\pi}{6}$, while in quadrant 2, $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Solving Trigonometric Equations

Example

Solve $3\tan x + 2 = 0$ on the interval $[0, 2\pi]$.

$$\begin{aligned} 3\tan x &= -2 \\ \tan x &= -\tfrac{2}{3} \\ x &= \tan^{-1}\left(-\tfrac{2}{3}\right) \end{aligned}$$

Tangent is negative in quadrants 2 and 4.

Calculating $\tan^{-1}\left(-\frac{2}{3}\right)$ yields approximately -0.588, which is a negative rotation into quadrant 4.

In quadrant 2, $x \approx \pi - 0.588 \approx 2.55$, while in quadrant 4, $x \approx 2\pi - 0.588 \approx 5.70$.

Solving Trigonometric Equations

Example

Solve $4\sin^2 x - 3 = 0$ on the interval $[0, 2\pi]$.

Factor as a difference of squares.

$$\begin{aligned} (2\sin x - \sqrt{3})(2\sin x + \sqrt{3}) &= 0 \\ x &= \sin^{-1}\left(\tfrac{\sqrt{3}}{2}\right) \text{ or } \sin^{-1}\left(-\tfrac{\sqrt{3}}{2}\right) \end{aligned}$$

In the case of $x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$, sine is positive in quadrants 1 and 2, so $x = \frac{\pi}{3}$ or $x = \frac{2\pi}{3}$.

In the case of $x = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, sine is negative in quadrants 3 and 4, so $x = \frac{4\pi}{3}$ or $x = \frac{5\pi}{3}$.

All four solutions are possible.

Solving Trigonometric Equations

Example

Solve $5\cos^2 x - 17\cos x + 6 = 0$ on the interval $[0, 2\pi]$.

Factor as a complex trinomial.

$$\begin{aligned} (5\cos x - 2)(\cos x - 3) &= 0 \\ x &= \cos^{-1}\left(\tfrac{2}{5}\right) \text{ or } \cos^{-1}(3) \end{aligned}$$

In the case of $x = \cos^{-1}\left(\frac{2}{5}\right)$, cosine is positive in quadrants 1 and 4, so $x \approx 1.16$ or $x \approx 2\pi - 1.16 \approx 5.12$.

The case of $x = \cos^{-1}(3)$ is not possible, since cosine has a range of $[-1, 1]$.

Thus, only two solutions are admissible.

Solving Trigonometric Equations

Example

Solve $25\tan^2 x - 90\tan x = -81$ on the interval $[0, 2\pi]$.

Rewrite as $25\tan^2 x - 90\tan x + 81 = 0$ and factor.

$$\begin{aligned}(5\tan x - 9)^2 &= 0\\ x &= \tan^{-1}\left(\tfrac{9}{5}\right)\end{aligned}$$

Since tangent is positive in quadrants 1 and 3, $x \approx 1.06$ or $x \approx \pi + 1.06 \approx 4.21$.

Solving Trigonometric Equations

Example

Solve $2\sin^3 x + 9\sin^2 x + 4\sin x$ on the interval $[0, \pi]$.

Common factor, then decompose.

$$\begin{aligned}(\sin x)(2\sin x + 1)(\sin x + 4) &= 0\\ x &= \sin^{-1}(0), \sin^{-1}\left(-\tfrac{1}{2}\right) \text{ or }\\ &\quad \sin^{-1}(-4)\end{aligned}$$

When $x = \sin^{-1}(0)$, $x = 0$ or π, both of which are on $[0, \pi]$.

When $x = \sin^{-1}\left(-\frac{1}{2}\right)$, $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$, neither of which is on $[0, \pi]$. The case where $x = \sin^{-1}(-4)$ is impossible.

So the only two valid solutions are $x = 0$ and $x = \pi$.

Solving Trigonometric Equations

Example

Solve $3\sec^2 x + 14\sec x - 5 = 0$ on the interval $[0, \pi]$.

$$\begin{aligned}(3\sec x - 1)(\sec x + 5) &= 0\\ x &= \sec^{-1}\left(\tfrac{1}{3}\right) \text{ or } \sec^{-1}(-5)\end{aligned}$$

Since secant is the reciprocal of cosine, $x = \cos^{-1}(3)$ (impossible) or $x = \cos^{-1}\left(-\frac{1}{5}\right)$.

Cosine is negative in quadrants 2 and 3, but quadrant 3 is not on $[0, \pi]$.

Thus, the only solution is $x \approx 1.77$.

Solving Trigonometric Equations

Example

Solve $3\sin 2x - 2 = 0$ on the interval $[0, 2\pi]$.

$$\begin{aligned}\sin 2x &= \tfrac{2}{3}\\ 2x &= \sin^{-1}\left(\tfrac{2}{3}\right)\end{aligned}$$

Since sine is positive in quadrants 1 and 2, it appears that there are two possible solutions on $[0, \pi]$.

Since $f(x) = 3\sin 2x - 2$ has a period of π, however, a value at x will have an image at $x + \pi$, falling on $[\pi, 2\pi]$.

This means that there are *four* solutions on $[0, 2\pi]$.

Solving Trigonometric Equations

When $2x = \sin^{-1}\left(\frac{2}{3}\right)$ is in quadrant 1, $2x \approx 0.730$, so $x \approx 0.365$.

When $2x = \sin^{-1}\left(\frac{2}{3}\right)$ is in quadrant 2, $2x \approx \pi - 0.730 \approx 2.412$, so $x \approx 1.206$.

If $x \approx 0.365$, then its image is at $0.365 + \pi \approx 3.507$.

If $x \approx 1.206$, then its image is at $1.206 + \pi \approx 4.348$.

Thus, the four solutions are approximately 0.365, 1.206, 3.507 and 4.348.

Questions?