MHF4U: Advanced Functions

Solving Trigonometric Equations Using Identities

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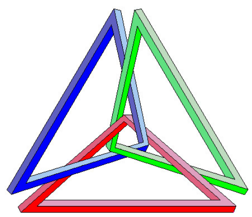


Solving Trigonometric Equations

Example

Solve $2\cos^2 x + \sin x - 1 = 0$ on $[0, 2\pi]$.

Since the equation involves both sine and cosine, use the Pythagorean Identity to express $\cos^2 x$ in terms of $\sin^2 x$ instead.

$$\begin{aligned} 2(1-\sin^2 x) + \sin x - 1 &= 0 \\ 2 - 2\sin^2 x + \sin x - 1 &= 0 \\ 2\sin^2 x - \sin x - 1 &= 0 \end{aligned}$$

Solving Trigonometric Equations

Factor the equation to find the zeroes.

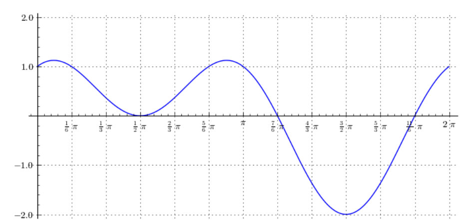
$$\begin{aligned} (\sin x - 1)(2\sin x + 1) &= 0 \\ \sin x &= 1, -\tfrac{1}{2} \end{aligned}$$

When $\sin x = 1$, $x = \frac{\pi}{2}$.

When $\sin x = -\frac{1}{2}$, $x = \frac{7\pi}{6}, \frac{11\pi}{6}$.

Thus, the three solutions are $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

Solving Trigonometric Equations



Solving Trigonometric Equations

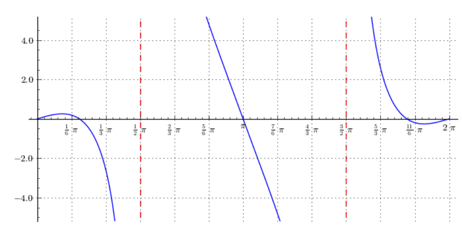
Example

Solve $5\sin x - 4\tan x = 0$ on $[0, 2\pi]$.

$$\begin{aligned} 5\sin x - 4\tan x &= 0 \\ 5\sin x &= 4\tan x \\ \frac{\sin x}{\tan x} &= \frac{4}{5} \\ \cos x &= \frac{4}{5} \\ x &= \cos^{-1}\left(\tfrac{4}{5}\right) \\ x &\approx 0.6435, 5.6397 \end{aligned}$$

Therefore, the two solutions are $x = 0.6435, 5.6397$.

Solving Trigonometric Equations



Wait, what?

The graph shows that there are 5 solutions, so where did the other three go?

Solving Trigonometric Equations

By replacing $\tan x$ with $\frac{\sin x}{\cos x}$, we inadvertently threw away three solutions.

Here is a better solution that uses the fact that $\sin x = \cos x \tan x$.

$$5\cos x \tan x - 4\tan x = 0$$
$$\tan x(5\cos x - 4) = 0$$

Thus, there are solutions when $\tan x = 0$, so $x = 0, \pi, 2\pi$.

The other two solutions, when $5\cos x - 4 = 0$, are the ones found earlier, $x \approx 0.6435, 5.6397$.

Therefore, the five solutions are $x = 0, \pi, 2\pi$ and $x \approx 0.6435, 5.6397$.

Solving Trigonometric Equations

Example

Solve $3\sin x - \cot x = 0$ on $[0, 2\pi]$.

Use the fact that $\cot x = \frac{\cos x}{\sin x}$, and apply the Pythagorean Identity.

$$3\sin x - \frac{\cos x}{\sin x} = 0$$
$$3(1 - \cos^2 x) - \cos x = 0$$
$$3\cos^2 x + \cos x - 3 = 0$$

Solving Trigonometric Equations

Use the quadratic formula to find the zeroes.

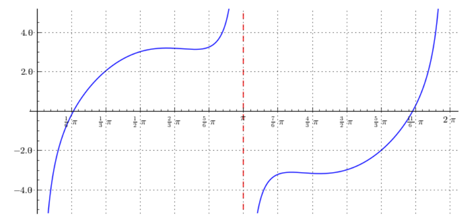
$$\cos x = \frac{-1 \pm \sqrt{1^2 - 4(3)(-3)}}{2(3)}$$
$$\cos x = \frac{-1 \pm \sqrt{37}}{6}$$

When $\cos x = \frac{-1+\sqrt{37}}{6}$, $x \approx 0.56, 5.72$.

$\cos x \neq \frac{-1-\sqrt{37}}{6}$, since $\frac{-1-\sqrt{37}}{6} \approx -1.18$, which is outside of the range of $\cos x$.

Thus, the two solutions are $x \approx 0.56, 5.72$.

Solving Trigonometric Equations



Solving Trigonometric Equations

Example

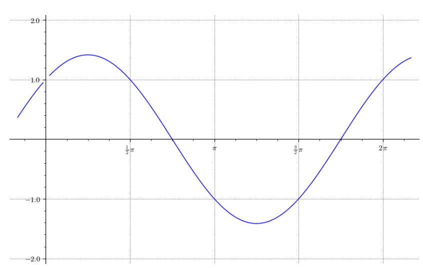
Solve $\sin x + \cos x = 1$ on $[0, 2\pi]$.

There is not much we can do here to isolate either $\sin x$ or $\cos x$, but what happens if we square both sides of the equation?

$$(\sin x + \cos x)^2 = 1^2$$
$$\sin^2 x + 2\sin x\cos x + \cos^2 x = 1$$
$$2\sin x\cos x + 1 = 1$$
$$\sin x\cos x = 0$$

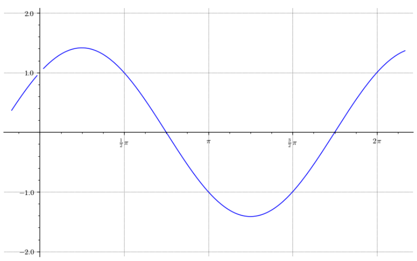
This implies that there are 5 solutions: when $\sin x = 0$, we get $x = 0, \pi, 2\pi$, and when $\cos x = 0$, we get $\frac{\pi}{2}, \frac{3\pi}{2}$.

Solving Trigonometric Equations



Checking the graph of $f(x) = \sin x + \cos x$, there are only 3 solutions where $f(x) = 1$.

Solving Trigonometric Equations



Checking the graph of $f(x) = \sin x + \cos x$, there are only 3 solutions ($x = 0, \frac{\pi}{2}, 2\pi$), where $f(x) = 1$ on the interval $[0, 2\pi]$.

Questions?

Some operations (like squaring) are *irreversible*. They can have effects, such as changing a sign, that cannot be undone.

Sometimes this results in *extraneous solutions*. These are solutions that result from algebraically manipulating an equation, but are not actually part of the original problem.

To remedy this, is it sometimes necessary to check all solutions to see if they are, indeed, relevant to the question.

In the case of $x = \frac{\pi}{2}$, $\sin \pi + \cos \pi = 0 + (-1) = -1$, so we reject this solution.

The same applies to $x = \frac{3\pi}{2}$, since $\sin \frac{3\pi}{2} + \cos \frac{3\pi}{2} = -1 + 0 = -1$.

Questions?