MHF4U: Advanced Functions

Remainder Theorem

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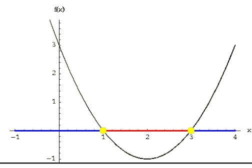

Remainder Theorem

Recap

Divide $f(x) = 2x^3 - 27x - 5$ by $x - 4$ and express the result in the forms $\frac{f(x)}{x-b} = Q + \frac{R}{x-b}$ and $f(x) = (x-b)Q + R$.

	2	0	−27	−5
4		8	32	20
	2	8	5	15

Therefore, $\frac{2x^3 - 27x - 5}{x-4} = 2x^2 + 8x + 5 + \frac{15}{x-4}$, or $2x^3 - 27x - 5 = (x-4)(2x^2 + 8x + 5) + 15$.

Remainder Theorem

What is the remainder when $f(x) = x^3 - 2x^2 + 4x - 1$ is divided by $x - 2$?

	1	−2	4	−1
2		2	0	8
	1	0	4	7

When $f(x)$ is divided by $x - 2$, the remainder is 7.

At the same time, $f(2) = 2^3 - 2(2)^2 + 4(2) - 1 = 7$.

Remainder Theorem

What is the remainder when $x^4 - 6x + 3$ is divided by $x + 1$?

	1	0	0	−6	3
−1		−1	1	−1	7
	1	−1	1	−7	10

When $f(x)$ is divided by $x + 1$, the remainder is 10.

Note that $f(-1) = (-1)^4 - 6(-1) + 3 = 10$.

In each case, when $f(x)$ is divided by $x - b$ it produces a remainder R, and $f(b) = R$.

This property is known as the *Remainder Theorem*.

Remainder Theorem

Remainder Theorem for Polynomials

If polynomial $P(x)$ is divided by $x - b$, then the linear remainder is $P(b)$. If $P(x)$ is divided by $ax - b$, the remainder is $P\left(\frac{b}{a}\right)$.

The Remainder Theorem allows us to determine the value of the linear remainder of a division without using synthetic or long division.

Remainder Theorem

Example

Determine the remainder when $f(x) = x^4 - 2x^2 - 5x$ is divided by $x - 3$. Verify using synthetic division.

$f(3) = 3^4 - 2(3)^2 - 5(3) = 48$.

	1	0	−2	−5	0
3		3	9	21	48
	1	3	7	16	48

Remainder Theorem

Example

Determine the remainder when $g(x) = 2x^4 - 3x^3 + 7x^2 - 11x + 5$ is divided by $2x - 1$. Verify using synthetic division.

$g\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^4 - 3\left(\frac{1}{2}\right)^3 + 7\left(\frac{1}{2}\right)^2 - 11\left(\frac{1}{2}\right) + 5 = 1.$

	2	−3	7	−11	5
$\frac{1}{2}$		1	−1	3	−4
	2	−2	6	−8	1

Remainder Theorem

Example

When $f(x) = 3x^3 + x^2 + kx + 7$ is divided by $x + 2$, the remainder is -21. Determine the value of k.

By the Remainder Theorem, $f(-2) = -21$.

Substitute $x = -2$ and $f(x) = -21$ into the equation, and solve for k.

$$\begin{aligned} -21 &= 3(-2)^3 + (-2)^2 + k(-2) + 7 \\ -21 &= -13 - 2k \\ 2k &= 8 \\ k &= 4 \end{aligned}$$

Remainder Theorem

Example

When $f(x) = px^3 - 6x^2 + qx - 10$ is divided by $x - 2$, the remainder is 12, and when it is divided by $x + 1$, the result is -24. Determine the values of p and q.

By the Remainder Theorem, $f(2) = 12$ and $f(-1) = -24$.

Use these values to set up the following linear system:

$$\begin{aligned} 12 &= p(2)^3 - 6(2)^2 + q(2) - 10 & \rightarrow \quad 4p + q &= 23 \\ -24 &= p(-1)^3 - 6(-1)^2 + q(-1) - 10 & \rightarrow \quad p + q &= 8 \end{aligned}$$

Using elimination, $3p = 15$, so $p = 5$.

Since $p + q = 8$, $q = 8 - 5 = 3$.

Remainder Theorem

Example

When $f(x) = 2x^3 + mx^2 - 4x + 11$ is divided by $x + 3$, the remainder is twice as great as when it is divided by $x - 1$. Determine the value of m.

By the Remainder Theorem, $f(1) = k$ and $f(-3) = 2k$ for some constant k.

Therefore, $f(-3) = 2f(1)$. Substitute, and solve for m.

$$\begin{aligned} 2(-3)^3 + m(-3)^2 - 4(-3) + 11 &= 2(2(1)^3 + m(1)^2 - 4(1) + 11) \\ 7m &= 49 \\ m &= 7 \end{aligned}$$

Questions?