MHF4U: Advanced Functions

Reciprocals of Linear Functions

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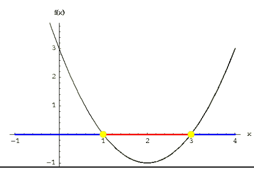


Reciprocals of Linear Functions

A linear function has the form $f(x) = kx + c$ for two real values k and c, such that k is the slope of the line and c is its y-intercept.

The reciprocal of a linear function has the form $f(x) = \dfrac{1}{kx+c}$, or $f(x) = \dfrac{1}{k\left(x+\frac{c}{k}\right)}$.

Additional transformations may be applied to change the graph's shape or position.

Asymptotes

The reciprocal of a linear function has two asymptotes: one vertical, and one horizontal.

The vertical asymptote (VA) occurs for the value of x that causes the denominator to equal zero.

Since we are looking for $x + \frac{c}{k} = 0$, the equation of the vertical asymptote is always $x = -\frac{c}{k}$.

For example, the function $f(x) = \dfrac{1}{5x-2}$ will have a VA with equation $x = \frac{2}{5}$.

Asymptotes

The equation of a horizontal asymptote (HA) can be found by dividing each term in a function by its highest power, then evaluating the function as $x \to \infty$.

Consider the function $f(x) = \dfrac{1}{5x-2}$.

$$\begin{aligned}\frac{1}{5x-2} &= \frac{\frac{1}{x}}{\frac{5x}{x} - \frac{2}{x}} \\ &= \frac{0}{5-0} \\ &= 0\end{aligned}$$

Thus, $f(x)$ has a HA with equation $f(x) = 0$. This is true for *all* functions of this form.

This technique will be handy later, so remember it.

Asymptotes

Example

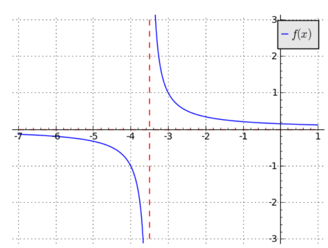
Determine the equations of the asymptotes for $f(x) = \frac{1}{2x+7}$, and state the domain and range.

The VA has equation $x = -\frac{7}{2}$.

The HA has equation $f(x) = 0$.

The domain is $\left(-\infty, -\frac{7}{2}\right) \cup \left(-\frac{7}{2}, \infty\right)$ and the range is $(-\infty, 0) \cup (0, \infty)$.

Asymptotes



Asymptotes

Example

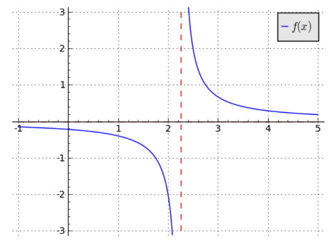
Determine the equations of the asymptotes for $f(x) = \frac{2}{4x-9}$, and state the domain and range.

The VA has equation $x = \frac{9}{4}$.

The HA has equation $f(x) = 0$. The 2 in the numerator stretches the graph, but does not change its location.

The domain is $\left(-\infty, \frac{9}{4}\right) \cup \left(\frac{9}{4}, \infty\right)$ and the range is $(-\infty, 0) \cup (0, \infty)$.

Asymptotes



Intercepts

As with any function, $f(x)$-intercepts can be found by setting $x = 0$ in its equation.

Since a function of the form $f(x) = \dfrac{a}{kx+c}$ has a horizontal asymptote at $f(x) = 0$, such functions have no x-intercepts.

Vertical translations will shift a graph and its horizontal asymptote, resulting in exactly one x-intercept.

Intercepts

Example

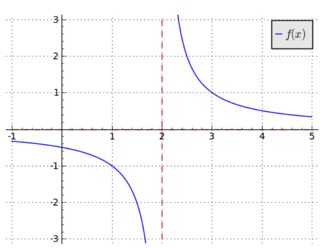
Determine the intercepts for $f(x) = \dfrac{1}{x-2}$.

Evaluate $f(0)$ to find the $f(x)$-intercept.

$$\begin{aligned} f(0) &= \frac{1}{0-2} \\ &= -\tfrac{1}{2} \end{aligned}$$

Since there are no vertical translations, there are no x-intercepts.

Intercepts



Intercepts

Example

Determine the intercepts for $f(x) = \dfrac{1}{x+5} + 2$.

Evaluate $f(0)$ to find the $f(x)$-intercept.

$$\begin{aligned} f(0) &= \frac{1}{0+5} + 2 \\ &= \tfrac{11}{5} \end{aligned}$$

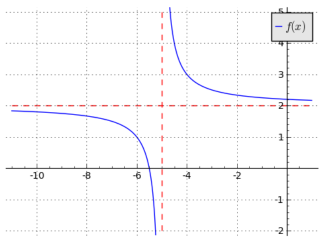
To determine the x-intercept, set $f(x) = 0$ and solve for x.

$$\begin{aligned} 0 &= \frac{1}{x+5} + 2 \\ -2(x+5) &= 1 \\ x+5 &= -\tfrac{1}{2} \\ x &= -\tfrac{11}{2} \end{aligned}$$

Intercepts

Since there has been a vertical translation of 2 units up, the horizontal asymptote has equation $f(x) = 2$.

There is a vertical asymptote at $x = -5$.



Intervals of Increase and Decrease

The graph of $f(x) = \dfrac{a}{kx + c}$ is a *hyperbola*, with two symmetric branches.

When $a > 0$, one branch is upper right and the other lower left.

If $a < 0$, there has been a vertical reflection and the branches are in the lower right and upper left.

Along with information about the asymptotes, this can be used to identify intervals of increase and decrease.

Intervals of Increase and Decrease

Example

Given $f(x) = -\dfrac{4}{x-3}$:

- Determine the equations of the asymptotes
- Determine the locations of the intercepts
- State the intervals on which the function is increasing or decreasing
- State the intervals on which the *slope* is increasing or decreasing

There are asymptotes at $x = 3$ and $f(x) = 0$.

The $f(x)$-intercept is at $\frac{4}{3}$. There is no x-intercept.

Intervals of Increase and Decrease

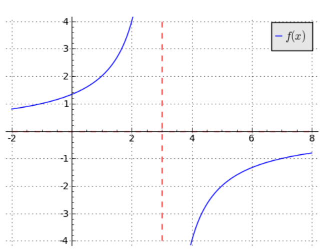
Since $f(x) = -4\left(\dfrac{1}{x-3}\right)$, $a < 0$, so the function has branches in the upper left and lower right.

As $x \to 3$ from the left, $f(x) \to \infty$. Therefore, $f(x)$ is increasing on $(-\infty, 3)$.

As $x \to \infty$, $f(x) \to 0$ from below. Therefore, $f(x)$ is increasing on $(3, \infty)$.

In general, the reciprocal of a linear function is *always* increasing when $a < 0$, and always decreasing when $a > 0$.

Intervals of Increase and Decrease



Intervals of Increase and Decrease

Since the left branch becomes steeper upwards as we move from left to right, the slope is increasing as $x \to 3$ from the left.

Since the right branch becomes less steep as we move from left to right, the slope is decreasing as $x \to \infty$.

Therefore, the slope is increasing on $(-\infty, 3)$ and decreasing on $(3, \infty)$.

It is important to remember the difference between the function increasing, and its slope increasing, as they are often different.

Questions?