MHF4U: Advanced Functions

Characteristics of Polynomial Functions

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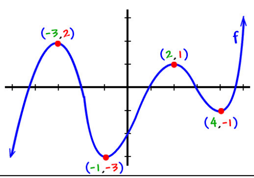
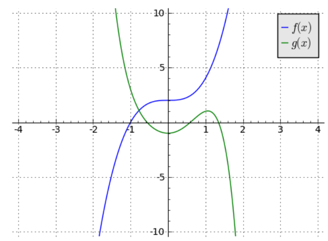


Odd-Degree Polynomial Functions

Consider the graphs of the polynomial functions $f(x) = 2x^3 + 2$ and $g(x) = -x^5 + 3x^2 - 1$ below.



What information can we obtain about the end behaviour, and the number of x-intercepts?

Odd-Degree Polynomial Functions

Since the leading coefficient of $f(x) = 2x^3 + 2$ is positive, and $f(x)$ is of odd degree, the end behaviour is from Q3-Q1.

The leading coefficient of $g(x) = -x^5 + 3x^2 - 1$ is negative, and $g(x)$ is also of odd degree, so the end behaviour is from Q2-Q4.

Like the simpler power functions, all odd-degree polynomials have Q3-Q1 or Q2-Q4 end behaviour, depending on the sign of the leading coefficient.

Odd-Degree Polynomial Functions

The range of all odd-degree polynomial functions is $(-\infty, \infty)$, so the graphs must cross the x-axis at least once.

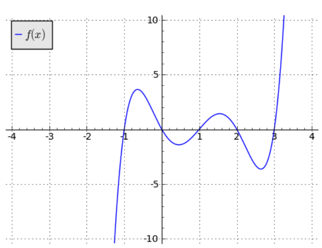
The graph of $f(x)$ has one x-intercept at $x = -1$.

Other graphs, such as that of $g(x)$, have more than one x-intercept.

Is there a limit on the number of x-intercepts an odd-degree polynomial function can have?

Odd-Degree Polynomial Functions

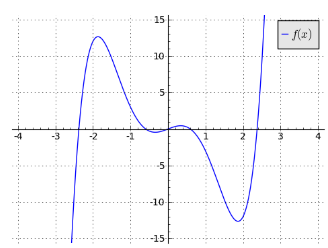
The graph of $f(x) = x^5 - 5x^4 + 5x^3 + 5x^2 - 6x$ has degree 5, and there are 5 x-intercepts.



In general, an odd-degree polynomial function of degree n may have up to n x-intercepts.

Odd-Degree Polynomial Functions

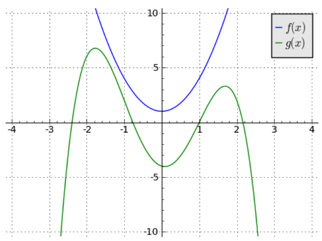
The graph of $f(x) = x^5 - 6x^3 + 2x$ has point symmetry about the origin.



Note how all exponents of the polynomial function are odd. Such a polynomial function is called an *odd function*.

Even-Degree Polynomial Functions

Consider the graphs of the polynomial functions $f(x) = 3x^2 + 1$ and $g(x) = -x^4 + 6x^2 - x - 4$ below.



Even-degree polynomial functions, such as $f(x)$ and $g(x)$, of degree n can have between 0 and n x-intercepts.

Even-Degree Polynomial Functions

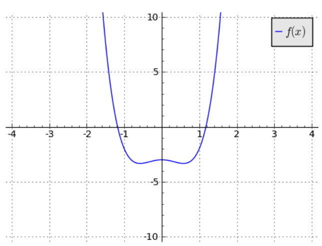
Since the leading coefficient of $f(x) = 3x^2 + 1$ is positive, the end behaviour is from Q2-Q1.

The leading coefficient of $g(x) = -x^4 + 6x^2 - x - 4$ is negative, and the end behaviour is from Q3-Q4.

Again, the end behaviour of even-degree polynomial functions is similar to power functions, and depends on the sign of the leading coefficient.

Even-Degree Polynomial Functions

The graph of $f(x) = 3x^4 - 2x^2 - 3$ is symmetric in the $f(x)$-axis.

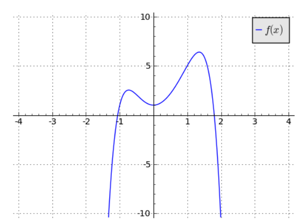


Note how all exponents of the polynomial function are even. Such a polynomial function is called an *even function*.

Classifying Polynomial Functions

Example

Classify $f(x)$ as even, odd or neither.



Although the function extends from Q3-Q4, it is not symmetric in the $f(x)$-axis, so it is neither even nor odd.

Classifying Polynomial Functions

Example

A polynomial function has x-intercepts at 0, 2, -3 and 7. What is the minimum degree of the polynomial if it is even? If it is odd?

Since there are four distinct x-intercepts, the minimum degree of the polynomial is 4, in the case where the degree is even.

If the function has odd degree, the minimum degree must be 5, since a cubic function (degree 3) can have at most 3 x-intercepts.

Finite Differences

Given a table of values for a polynomial function, it is possible to determine the value of the leading coefficient.

First, let's define $n!$ ("n factorial") as follows:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

Some examples:

$$3! = 3 \times 2 \times 1 = 6$$

$$10! = 10 \times 9 \times \ldots \times 2 \times 1 = 3\,628\,800$$

Finite Differences

Let's examine the finite differences for $f(x) = x^3$.

x	$f(x)$	$\Delta 1$	$\Delta 2$	$\Delta 3$
0	0			
1	1	1		
2	8	7	6	
3	27	19	12	6
4	64	37	18	6
5	125	61	24	6

Note that $f(x)$ has degree 3 and the third differences are constant.

Also note that $3! = 6$, which is the value of the third differences.

Finite Differences

Now let's examine the finite differences for $f(x) = -2x^4$.

x	$f(x)$	$\Delta 1$	$\Delta 2$	$\Delta 3$	$\Delta 4$
0	0				
1	-2	-2			
2	-32	-30	-28		
3	-162	-130	-100	-72	
4	-512	-350	-220	-120	-48
5	-1250	-738	-388	-168	-48

$f(x)$ has degree 4 and the fourth differences are constant.

$4! = 24$, which is *not* the value of the fourth differences.

However, multiplying $4!$ by the leading coefficient, -2, *does* produce the value -48.

Finite Differences

Finite Differences for Polynomial Functions

Given a polynomial function with degree n and leading coefficient a, the nth finite differences are constant, with a value of $an!$.

We can use this relationship to determine the leading coefficient of any polynomial function, given enough points on its graph.

Finite Differences

Example

A polynomial function has constant fifth differences with a value of -360. Determine the degree of the function, and its leading coefficient.

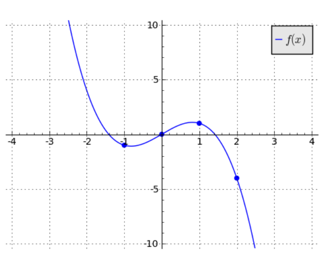
Since the fifth differences are constant, the degree is 5.

Rearranging $a \times 5! = -360$ to solve for a, we obtain $a = \frac{-360}{5!} = \frac{-360}{120} = -3$.

Finite Differences

Example

Determine the leading coefficient of the cubic function below.



Four successive points are $(-1,-1)$,$(0,0)$,$(1,1)$ and $(2,-4)$.

Finite Differences

Construct a table of finite differences.

x	$f(x)$	$\Delta 1$	$\Delta 2$	$\Delta 3$
-1	-1			
0	0	1		
1	1	1	0	
2	-4	-5	-6	-6

Rearranging $a \times 3! = -6$ to solve for a, we obtain $a = \frac{-6}{3!} = \frac{-6}{6} = -1$.

Questions?