MHF4U: Advanced Functions

Oblique Asymptotes

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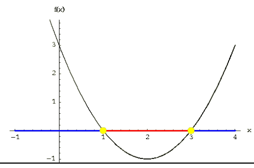


Oblique Asymptotes

In addition to horizontal and vertical asymptotes, a function may have *oblique* asymptotes.

Oblique asymptotes are sometimes called "slant" asymptotes because they have the form $y = ax + b$, where $a \neq 0$.

A function will have an oblique asymptote if the degree of the numerator is one greater than that of the denominator.

A function will never have both oblique and horizontal asymptotes.

Oblique Asymptotes

Example

Determine the equation of the oblique asymptote for $f(x) = \frac{x^2 - 1}{x}$, and graph the function.

Use long division to determine the equation of the oblique asymptote.

$$\begin{array}{r|l} & x \\ \hline x) & x^2 - 1 \\ & -x^2 \\ \hline \end{array}$$

The equation of the oblique asymptote is $y = x$.

Oblique Asymptotes

There is a vertical asymptote at $x = 0$, as given by the denominator.

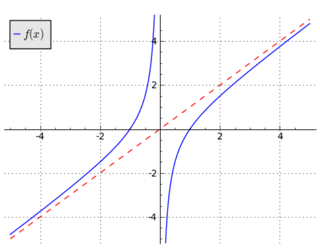
There is no $f(x)$-intercept, since setting $x = 0$ causes a division by zero error.

Since $x^2 - 1$ is a difference of squares, $f(x)$ has x-intercepts at ± 1.

These points, along with the asymptotes, give us enough information to accurately sketch the graph, but we can test values of x close to the vertical asymptote to get a better picture.

As $x \to 0$ from the left, $f(x) \to \infty$, and as $x \to 0$ from the right, $f(x) \to -\infty$.

Oblique Asymptotes



Oblique Asymptotes

Example

Graph $f(x) = \frac{x^2 + 2x - 3}{x + 1}$.

$$\begin{array}{r|l} & x + 1 \\ \hline x+1) & x^2 + 2x - 3 \\ & -x^2 - x \\ \hline & x - 3 \\ & -x - 1 \\ \hline & -4 \end{array}$$

There is an oblique asymptote with equation $y = x + 1$.

Oblique Asymptotes

There is a vertical asymptote at $x = -1$, since the denominator is zero when $x = -1$.

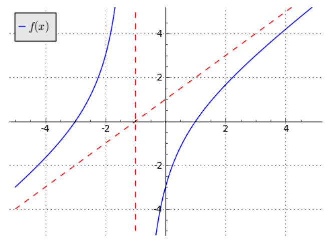
Setting $x = 0$ gives an $f(x)$-intercept of -3.

Since $f(x)$ factors as $f(x) = \dfrac{(x-1)(x+3)}{x+1}$, there are x-intercepts at 1 and -3.

As $x \to 0^-$, $f(x) \to \infty$, and as $x \to 0^+$, $f(x) \to -\infty$.

In the notation above, $x \to k^-$ means "as x approaches k from the left", while $x \to k^+$ is from the right.

Oblique Asymptotes



Oblique Asymptotes

Example

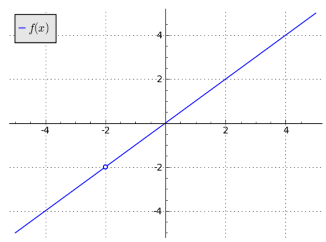
Graph $f(x) = \dfrac{x^2+2x}{x+2}$.

In this case, note that $f(x) = \dfrac{x(x+2)}{x+2} = x$, where $x \neq -2$.

Thus, the graph of $f(x)$ is the same as the graph of $y = x$, but with a point discontinuity at $x = -2$.

It is generally a good idea to check for the same root in both the numerator and denominator before doing any extra work.

Oblique Asymptotes



Oblique Asymptotes

A More Complex Example

Graph $f(x) = \dfrac{x^3-2x^2-x+2}{x^2-x-6}$.

Let $f(x) = \dfrac{g(x)}{h(x)}$.

By the FT, $g(1) = g(2) = g(-1) = 0$, and $h(-2) = h(3) = 0$.

Therefore, $f(x) = \dfrac{(x-1)(x-2)(x+1)}{(x+2)(x-3)}$.

There are vertical asymptotes at $x = -2$ and $x = 3$.

The x-intercepts are at 1, 2 and -1.

The $f(x)$-intercept is at $-\frac{1}{3}$.

Oblique Asymptotes

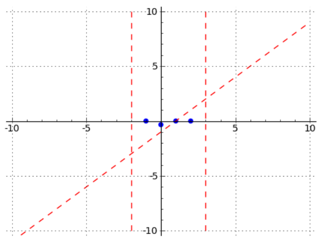
Use long division to determine the equation of the oblique asymptote.

$$\begin{array}{r|rrrr} & x & -1 & & \\ \hline x^2-x-6 & x^3 & -2x^2 & -x & +2 \\ & -x^3 & +x^2 & +6x & \\ \hline & & -x^2 & +5x & +2 \\ & & x^2 & -x & -6 \\ \hline & & & 4x & -4 \end{array}$$

There is an oblique asymptote with equation $y = x - 1$.

Oblique Asymptotes

Putting everything together produces the following graph.



Clearly, more information is needed for an accurate graph.

Oblique Asymptotes

Test points on either side of the vertical asymptotes.

As $x \to -2^-$, $f(x) \to -\infty$, so the function decreases.

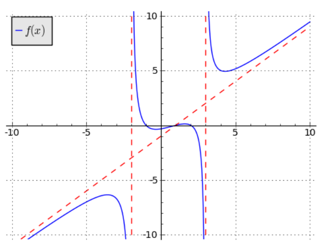
As $x \to -2^+$, $f(x) \to \infty$, so the function increases.

As $x \to 3^-$, $f(x) \to -\infty$, so the function decreases.

As $x \to 3^+$, $f(x) \to \infty$, so the function increases.

This gives us enough information to make a rough sketch.

Oblique Asymptotes



Note that $f(x)$ intersects the oblique asymptote at $x = 1$. Since asymptotes describe *end behaviour* of a function, this is perfectly normal.

Questions?