MHF4U: Advanced Functions

Factor Theorem

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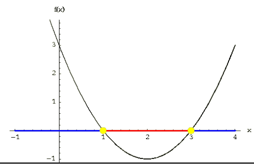

Factor Theorem

Example

Divide $f(x) = x^3 + 4x^2 + x - 6$ by $x - 1$.

	1	4	1	−6
1		1	5	6
	1	5	6	0

So $f(x) = (x-1)(x^2+5x+6)$, with no remainder.

Therefore, $x - 1$ is a factor of $f(x)$.

Factor Theorem

This application of the Remainder Theorem is called the *Factor Theorem*.

Factor Theorem (FT) for Polynomials

$x - b$ is a factor of polynomial $P(x)$ iff $P(b) = 0$, and $ax - b$ is a factor of $P(x)$ iff $P\left(\frac{b}{a}\right) = 0$.

When used with synthetic (or long) division, the FT provides a method of factoring polynomials of any degree, assuming they are factorable.

By determining values of x such that $P(x) = 0$, we can determine factors of $P(x)$, then perform a division to reduce the degree of the polynomial.

Factor Theorem

Example

Factor $f(x) = x^3 - 3x^2 - 50$.

Test values of x to find a factor of $f(x)$.

$f(1) = -52$, so $x - 1$ is not a factor.

$f(2) = -54$, so $x - 2$ is not a factor.

$f(3) = -50$, so $x - 3$ is not a factor.

$f(4) = -34$, so $x - 4$ is not a factor.

$f(5) = 0$, so $x - 5$ is a factor.

Factor Theorem

Use synthetic division to determine the quotient when $f(x) = x^3 - 3x^2 - 50$ is divided by $x - 5$.

	1	−3	0	−50
5		5	10	50
	1	2	10	0

So $f(x) = (x-5)(x^2+2x+10)$.

$x^2 + 2x + 10$ is not factorable, so $f(x)$ is fully factored.

Integral Zero Theorem

Guessing values for x may take a long time, so there are some ways in which we can reduce the sample space.

Consider a polynomial in both factored form, $f(x) = k(x-r_1)(x-r_2)...(x-r_n)$, and standard form, $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$.

Note that $a_0 = k \times r_1 \times r_2 \times \ldots \times r_n$.

Therefore, if $x - b$ is a factor of $f(x)$, it must be the case that b is a factor of a_0.

In the previous example, $x - 5$ was a factor of $f(x)$, and 5 is a factor of -50.

$x - 3$ was not a factor of $f(x)$, and 3 is not a factor of -50.

However, $x - 2$ was not a factor of $f(x)$, even though 2 is a factor of -50.

Integral Zero Theorem

This insight is known as the *Integral Zero Theorem*.

Integral Zero Theorem (IZT) for Polynomials

If $x - b$ is a factor of polynomial $P(x)$, with integer coefficients an leading coefficient 1, then b is a factor of the constant term of $P(x)$.

Factors may be either positive or negative.

As its name suggests, the IZT provides a method of finding all zeroes that can be expressed as integers.

Any non-integral solutions will not be found using the IZT, but it provides a good starting point.

Integral Zero Theorem

Example

Factor $f(x) = x^3 + 6x^2 - x - 30$.

Factors of -30 are ±1, ±2, ±3, ±5, ±6, ±10, ±15 and ±30.

Since $f(1) = -5$, $x - 1$ is not a factor of $f(x)$.

Since $f(2) = 0$, $x - 2$ is a factor of $f(x)$.

	1	6	−1	−30
2		2	16	30
	1	8	15	0

Therefore, $f(x) = (x - 2)(x^2 + 8x + 15)$.

Factoring the simple trinomial, $f(x) = (x - 2)(x + 3)(x + 5)$.

Rational Zero Theorem

The IZT may be extended to handle rational numbers, $\frac{b}{a}$.

Rational Zero Theorem (RZT) for Polynomials

If $P(x)$ is a polynomial with integer coefficients, and if $\frac{b}{a}$ is a rational zero of $P(x)$, then b is a factor of the constant term of $P(x)$ and a is a factor if the leading coefficient.

Again, factors may be either positive or negative.

While the RZT provides additional possible factors, it is generally it is a good idea to check for integers first.

Rational Zero Theorem

Example

List all possible rational zeroes for $f(x) = 2x^3 - 5x + 6$.

Factors of the constant term, 6, are ±1, ±2, ±3 and ±6.

Factors of the leading coefficient, 2, are ±1 and ±2.

By the RZT, the possible rational zeroes of $f(x)$ are ±1, ±2, ±3, ±6, $\pm\frac{1}{2}$ and $\pm\frac{3}{2}$.

Rational Zero Theorem

Example

Determine the real roots of $f(x) = 2x^3 - 3x^2 - 10x + 15$.

By the RZT, the possible rational zeroes of $f(x)$ are ±1, ±3, ±5, ±15, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{5}{2}$ and $\pm\frac{15}{2}$.

Since $f\left(\frac{3}{2}\right) = 0$, $2x - 3$ is a factor of $f(x)$.

	2	−3	−10	15
$\frac{3}{2}$		3	0	−15
	2	0	−10	0

Dividing the quotient by 2, $f(x) = (2x - 3)(x^2 - 5)$.

Since $x^2 - 5 = (x - \sqrt{5})(x + \sqrt{5})$, the real roots of $f(x)$ are $\frac{3}{2}$ and $\pm\sqrt{5}$.

Rational Zero Theorem

Example

Factor $f(x) = 4x^4 - 12x^3 - 17x^2 + 3x + 4$.

By the RZT, factors are ±1, ±2, ±4, $\pm\frac{1}{2}$ and $\pm\frac{1}{4}$.

Since $f(-1) = 0$, $x + 1$ is a factor of $f(x)$.

	4	−12	−17	3	4
−1		−4	16	1	−4
	4	−16	−1	4	0

Therefore, $f(x) = (x + 1)(4x^3 - 16x^2 - x + 4)$.

Rational Zero Theorem

To factor $4x^3 - 16x^2 - x + 4$ further, repeat the process.

Again, factors are ± 1, ± 2, ± 4, $\pm\frac{1}{2}$ and $\pm\frac{1}{4}$.

Since $f(4) = 0$, $x - 4$ is a factor of $f(x)$.

	4	−16	−1	4
4		16	0	−4
	4	0	−1	0

Therefore, $f(x) = (x+1)(x-4)(4x^2 - 1)$.

Since $4x^2 - 1$ is a difference of squares,
$f(x) = (x+1)(x-4)(2x-1)(2x+1)$.

Questions?