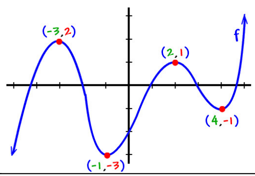
MHF4U: Advanced Functions

Equations and Graphs of Polynomial Functions

J. Garvin



Polynomial Functions In Factored Form

Polynomials are generally written in *standard form*, such as $f(x) = x^3 + 4x^2 + x - 6$.

A more useful way to write a polynomial function's equation is to use *factored form*, such as $f(x) = (x-1)(x+2)(x+3)$.

Each factor corresponds to an x-intercept of the function.

Factored Form of a Polynomial Function

An equation of a polynomial function is in factored form if it is written as $f(x) = a(x-r_1)(x-r_2)\ldots(x-r_n)$, where $(x-r_k)$ is a factor corresponding to x-intercept r_k.

Note that it is not always possible to express a polynomial function using factored form.

Polynomial Functions In Factored Form

Example

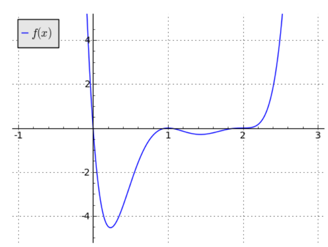
Identify the factors, and x-intercepts, of the polynomial function $f(x) = -2x(x+5)(x-3)$.

$f(x)$ has three factors: x, $x+5$ and $x-3$.

These factors correspond to x-intercepts 0, -5 and 3.

Order of a Factor

Examine the x-intercepts of $f(x) = 6x(x-1)^2(x-2)^3$ below.



How does the function behave around each x-intercept?

Order of a Factor

The function $f(x) = 6x(x-1)^2(x-2)^3$ has x-intercepts at 0, 1 and 2.

At $x = 0$, the function changes from positive to negative, passing through the x-axis.

The function remains negative on either side of $x = 1$, "bouncing" off of the x-axis.

At $x = 2$, the function changes from negative to positive, passing through the x-axis.

How does this behaviour relate to the factors of the function?

Order of a Factor

In two cases, $x = 0$ and $x = 2$, the exponents were odd. Both of these cases saw the function change from positive to negative, or vice versa.

In the other case, $x = 1$, the exponent was even. No change in sign occurred here.

Order of a Factor

The factor $(x-r)^n$ has *order n*. If n is odd, the function crosses the x-axis at r. If n is even, the function touches (but does not cross) the x-axis at r.

Used in conjunction with a function's end behaviour, identifying the order of each factor is a useful tool for sketching graphs.

Graphs of Polynomial Functions

Example

Sketch a graph of $f(x) = (x+3)^2(x-1)$.

$f(x)$ has two distinct x-intercepts, at $x = -3$ and $x = 1$.

Another way to write the equation is $f(x) = (x+3)(x+3)(x-1)$. Multiplying all terms containing x, we obtain x^3, so $f(x)$ has degree 3 (cubic).

The leading coefficient is positive, so $f(x)$ has Q3-Q1 end behaviour. Therefore, $f(x)$ is negative as $x \to -\infty$.

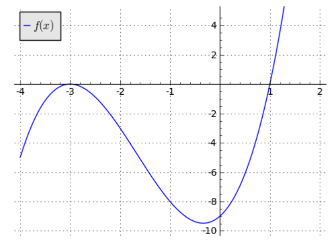
Moving from left to right, the first x-intercept is at $x = -3$, where it has order 2. Thus, the function touches the x-axis at $x = -3$, but stays negative beyond it.

The next x-intercept is at $x = 1$, where it has order 1. $f(x)$ changes from negative to positive at $x = 1$.

Order of a Factor

One final piece point is the y-intercept, which can be found by multiplying all of the constant terms and the leading coefficient.

In this case, the y-intercept is $(3)(3)(-1) = -9$.



Graphs of Polynomial Functions

Example

Sketch a graph of $f(x) = -2x(x+1)(x-2)^2$.

$f(x)$ has three distinct x-intercepts, at $x = -1$, $x = 0$ and $x = 2$.

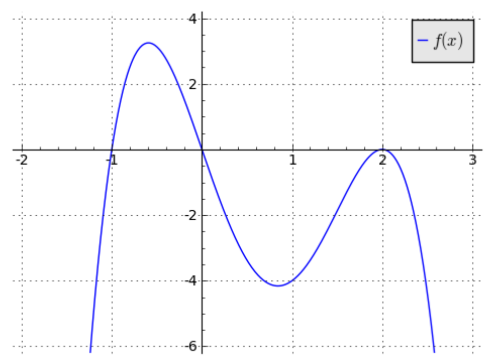
$f(x)$ is a quartic function, with degree 4.

The leading coefficient is negative, so $f(x)$ has Q3-Q4 end behaviour. $f(x)$ is negative as $x \to -\infty$.

From left to right, $f(x)$ changes from negative to positive at $x = -1$, changes from positive to negative at $x = 0$, and touches the x-axis at $x = 2$.

The y-intercept is $-2(0)(1)(-2)(-2) = 0$.

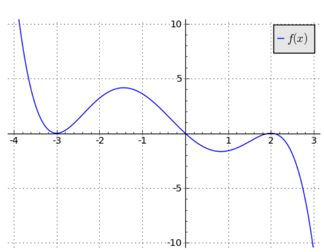
Graphs of Polynomial Functions



Graphs of Polynomial Functions

Example

Given the graph of $f(x)$ below, state the minimum possible degree, sign of the leading coefficient, factors, x-intercepts and intervals where the function is positive or negative.



Graphs of Polynomial Functions

$f(x)$ has Q2-Q4 end behaviour, so the degree must be odd and the leading coefficient is negative.

There are x-intercepts at $x = -3$ (even order), $x = 0$ (odd order) and $x = 2$ (even order), so the minimum degree is 5.

This is confirmed by the fact that there are 4 local minimums and maximums.

$f(x)$ is positive on the intervals $(-\infty, -3) \cup (-3, 0)$.

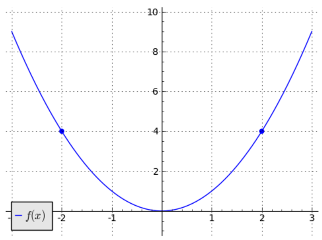
Since zero is neither positive nor negative, it is not included in the interval.

$f(x)$ is negative on the intervals $(0, 2) \cup (2, \infty)$.

Symmetry

Recall that an *even* function is symmetric in the $f(x)$-axis.

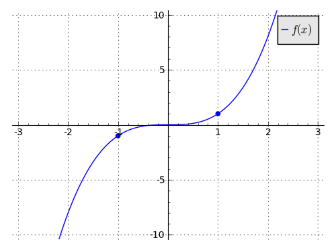
Any function that is symmetric in the $f(x)$-axis has the property that $f(x) = f(-x)$.



Symmetry

An *odd* function is point-symmetric about the origin.

Any function that has point symmetry about the origin has the property that $f(-x) = -f(x)$.



Symmetry

Example

Verify algebraically that $f(x) = 2x^4 + 3x^2 - 1$ is symmetric in the $f(x)$-axis.

$$\begin{aligned} f(-x) &= 2(-x)^4 + 3(-x)^2 - 1 \\ &= 2x^4 + 3x^2 - 1 \\ &= f(x) \end{aligned}$$

Therefore, $f(x)$ is symmetric in the $f(x)$-axis. It is an even function.

Symmetry

Example

Algebraically classify $f(x) = 2x^3 + x^2 - 5x$ as even, odd or neither.

Test if $f(x)$ is even first.

$$\begin{aligned} f(-x) &= 2(-x)^3 + (-x)^2 - 5(-x) \\ &= -2x^3 + x^2 + 5x \\ &\neq f(x) \end{aligned}$$

Therefore, $f(x)$ is not even.

Symmetry

Test if $f(x)$ is odd next.

$$\begin{aligned} f(-x) &= -2x^3 + x^2 + 5x \\ -f(x) &= -(2x^3 + x^2 - 5x) \\ &= -2x^3 - x^2 + 5x \\ f(-x) &\neq -f(x) \end{aligned}$$

Therefore, $f(x)$ is not odd either.

Questions?