MHF4U: Advanced Functions

Double and Half Angle Identities

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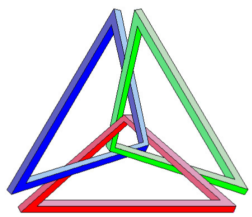

Double Angle Identities

Recall the addition identity for sine, $\sin(x+y) = \sin x \cos y + \sin y \cos x$.

Note that $\sin 2x = \sin(x+x)$.

Using the addition identity, $\sin(x+x) = \sin x \cos x + \sin x \cos x = 2 \sin x \cos x$.

Double Angle Identity for Sine

For any angle x, $\sin 2x = 2 \sin x \cos x$.

Double Angle Identities

Example

Verify that $\sin \frac{\pi}{3} = 2 \sin \frac{\pi}{6} \cos \frac{\pi}{6}$.

$$\sin \tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}$$

$$\begin{aligned} 2 \sin \tfrac{\pi}{6} \cos \tfrac{\pi}{6} &= 2 \cdot \tfrac{1}{2} \cdot \tfrac{\sqrt{3}}{2} \\ &= \tfrac{\sqrt{3}}{2} \end{aligned}$$

Double Angle Identities

We can also develop *several* double angle identities for cosine.

Recall that $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

Use $\cos 2x = \cos(x+x)$.

$\cos(x+x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$.

Using the Pyhtagorean Identity,
$\cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$, or
$\cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$.

Double Angle Identities for Cosine

For any angle x, $\cos 2x = \cos^2 x - \sin^2 x$, or $\cos 2x = 1 - 2\sin^2 x$, or $\cos 2x = 2\cos^2 x - 1$.

Double Angle Identities

Example

Verify that $\cos \frac{\pi}{3} = \cos^2 \frac{\pi}{6} - \sin^2 \frac{\pi}{6}$.

$$\cos \tfrac{\pi}{3} = \tfrac{1}{2}$$

$$\begin{aligned} \cos^2 \tfrac{\pi}{6} - \sin^2 \tfrac{\pi}{6} &= \left(\tfrac{\sqrt{3}}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 \\ &= \tfrac{3}{4} - \tfrac{1}{4} \\ &= \tfrac{1}{2} \end{aligned}$$

Triple Angle Identities, and Beyond

Using the double angle identities, and the addition identities for sine and cosine, we can extend the same derivation to handle cases like $\sin 3x$, or $\cos 4x$, etc.

Example

Prove that $\sin 3x = (\sin x)(4\cos^2 x - 1)$.

$$\begin{aligned} \sin 3x &= \sin(2x + x) \\ &= \sin 2x \cos x + \sin x \cos 2x \\ &= 2 \sin x \cos x \cos x + (\sin x)(2\cos^2 x - 1) \\ &= 2 \sin x \cos^2 x + 2 \sin x \cos^2 x - \sin x \\ &= 4 \sin x \cos^2 x - \sin x \\ &= (\sin x)(4\cos^2 x - 1) \end{aligned}$$

Half Angle Identities

Using the double angle identities, it is possible to derive half angle identities for both sine and cosine.

Since $\cos 2x = 1 - 2\sin^2 x$, it is possible to solve for $\sin x$.

$$\begin{aligned}\cos 2x - 1 &= -2\sin^2 x\\ \sin^2 x &= \frac{1-\cos 2x}{2}\\ \sin x &= \pm\sqrt{\frac{1-\cos 2x}{2}}\end{aligned}$$

If $a = 2x$, then $x = \frac{a}{2}$, and $\sin\frac{a}{2} = \pm\sqrt{\frac{1-\cos a}{2}}$.

The sign depends on the quadrant in which the angle lies.

Half Angle Identities

Similarly, $\cos 2x = 2\cos^2 x - 1$, and solving for $\cos x$ yields $\cos x = \pm\sqrt{\frac{1+\cos 2x}{2}}$.

If $a = 2x$, then $x = \frac{a}{2}$, and $\cos\frac{a}{2} = \pm\sqrt{\frac{1+\cos a}{2}}$.

Half Angle Identities

For any angle x, $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$ and $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$.

Half Angle Identities

Example

Determine an exact value for $\sin\frac{\pi}{8}$.

Note that $\sin\frac{\pi}{8} = \frac{1}{2}\cdot\frac{\pi}{4}$, and lies in quadrant 1.

$$\begin{aligned}\sin\frac{\pi}{8} &= \sqrt{\frac{1-\cos\frac{\pi}{4}}{2}}\\ &= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}\\ &= \sqrt{\frac{2-\sqrt{2}}{4}}\\ &= \frac{\sqrt{2-\sqrt{2}}}{2}\end{aligned}$$

Questions?