MHF4U: Advanced Functions

Rate of Change, Part 1

Average Rate of Change

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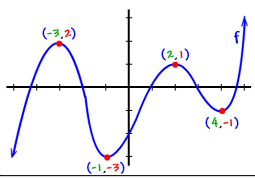


Rate of Change

A *rate of change* is a change in one quantity with respect to another.

Rates of change can be measured in two ways:

- an *average* rate of change is measured over an interval (e.g. the speed of a car over 3 hours)
- an *instantaneous* rate of change is measured at a given instant (e.g. the car's speed at exactly 1 hour, 40 minutes)

Instantaneous rate of change has strong connections with calculus, and will be investigated more thoroughly then.

Rates of Change

Example

A car travels 260 km from Toronto to Kingston in 3 hours. What is the average speed of the car, in km/h?

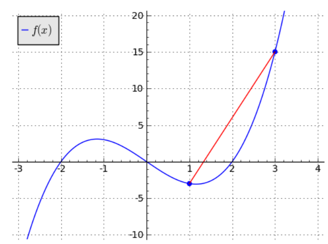
The speed of the car is given by the change in distance (km), divided by the change in time (h).

$$\frac{\Delta d}{\Delta t} = \frac{\text{change in distance}}{\text{change in time}}$$
$$= \frac{260}{3}$$
$$= \approx 87 \text{ km/h}$$

The average speed of the car is approximately 87 km/h.

Average Rate of Change

On a graph, the average rate of change is represented by a *secant* line from $(a, f(a))$ to $(b, f(b))$.



Average Rate of Change

The slope of this line can be found by calculating the ratio of the rise, $f(b) - f(a)$, to the run, $b - a$.

Average Rate of Change

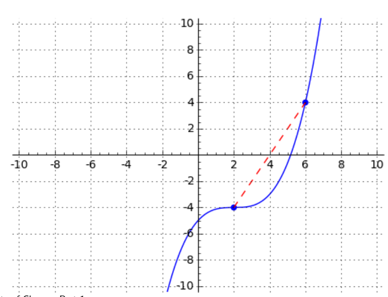
The average rate of change on $[a, b]$ can be calculated by $\frac{\Delta f(x)}{\Delta x} = \frac{f(b) - f(a)}{b - a}$.

Note that this is just the familiar "slope formula" used in earlier grades.

Average Rate of Change

Example

Calculate the slope of the secant connecting $(2, -4)$ and $(6, 4)$ on the graph below.



Average Rate of Change

Use the formula $\frac{\Delta f(x)}{\Delta x} = \frac{f(b)-f(a)}{b-a}$, where $a = 2$, $f(a) = -4$, $b = 6$ and $f(b) = 4$.

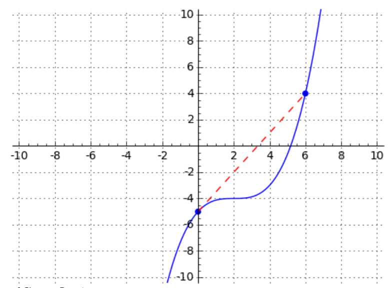
$$\begin{aligned}\frac{\Delta f(x)}{\Delta x} &= \frac{4-(-4)}{6-2} \\ &= \frac{8}{4} \\ &= 2\end{aligned}$$

The value of the average rate of change is 2.

Average Rate of Change

Example

On the same graph, calculate the slope of the secant connecting $(0, -5)$ and $(6, 4)$.



Average Rate of Change

Use the formula $\frac{\Delta f(x)}{\Delta x} = \frac{f(b)-f(a)}{b-a}$, where $a = 0$, $f(a) = -5$, $b = 6$ and $f(b) = 4$.

$$\begin{aligned}\frac{\Delta f(x)}{\Delta x} &= \frac{4-(-5)}{6-0} \\ &= \tfrac{9}{6} \\ &= \tfrac{3}{2}\end{aligned}$$

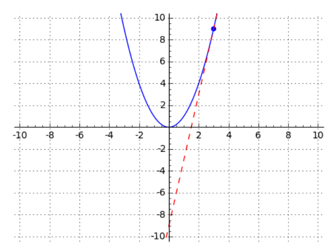
The value of the average rate of change is $\frac{3}{2}$.

A function's average rate of change depends on what interval is used.

Average Rate of Change

Example

A tangent to the curve $f(x) = x^2$ at $x = 3$ has a slope of 6. Calculate the slopes of the secants on the intervals $[0, 3]$, $[1, 3]$ and $[2, 3]$, and compare them to that of the tangent.



Average Rate of Change

When $x = 0$, $f(0) = 0$, and when $x = 3$, $f(3) = 9$. The slope of the secant from $(0, 0)$ to $(3, 9)$ is

$$\begin{aligned}\frac{\Delta f(x)}{\Delta x} &= \frac{9-0}{3-0} \\ &= 3\end{aligned}$$

When $x = 1$, $f(1) = 1$. The slope from $(1, 1)$ to $(3, 9)$ is

$$\begin{aligned}\frac{\Delta f(x)}{\Delta x} &= \frac{9-1}{3-1} \\ &= 4\end{aligned}$$

When $x = 2$, $f(2) = 4$. The slope from $(2, 4)$ to $(3, 9)$ is

$$\begin{aligned}\frac{\Delta f(x)}{\Delta x} &= \frac{9-4}{3-2} \\ &= 5\end{aligned}$$

Average Rate of Change

As the width of an interval decreases, the slope of a secant becomes more representative of the rate of change for a curve.

If the interval was sufficiently small, the slope of the secant would be very close to the slope of the *tangent* at a given point.

The slope of a tangent will be explored in more detail in the next lesson.

Average Rate of Change

Example

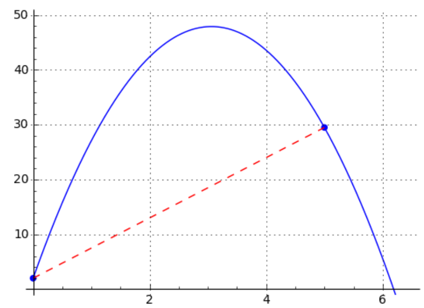
A projectile's height, h metres, is given by the equation $h(t) = -4.9t^2 + 30t + 2$, where t is the time in seconds. What is the rate of change of the projectile's height during the first 5 seconds of flight?

Evaluate the function at the ends of the interval $[0, 5]$.

$$\begin{aligned} h(0) &= -4.9(0)^2 + 30(0) + 2 \\ &= 2 \\ h(5) &= -4.9(5)^2 + 30(5) + 2 \\ &= 29.5 \end{aligned}$$

The rate of change of height is $\frac{29.5-2}{5-0} = 5.5$ m/s.

Average Rate of Change



Questions?