MHF4U: Advanced Functions
Review of Factoring Techniques

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1 What is Factoring?

Factoring is the process of writing an algebraic expression or equation as the product of two or more *factors* – simpler expressions that multiply to produce the original.

For example, the equation $x^2 + 5x + 6 = 0$ factors as:

$(x+2)(x+3)$

The roots (or *zeroes*) of an equation are the values that satisfy the equation. They are closely related to the factors. The equation $x^2+5x+6 = 0$ has two real roots:

$x = -2$
$x = -3$

NOTE: SAME VALUES, DIFFERENT SIGNS

Factoring is typically performed using integral constants and coefficients, although the roots themselves may be rational or irrational numbers.

Graphically, the roots of a function correspond to its x-intercepts.

2 Common Factoring

The simplest method of factoring involves identifying the Greatest Common Factor (GCF) of all terms.

$2 < 3$

For example, the GCF for the expression $6x^3 - 10x^2$ is:

$2x^2$

6 : 1, $\boxed{2}$, 3, 6
10 : 1, $\boxed{2}$, 5, 10

Factoring this out from both terms, we obtain:

$2x^2(3x - 5)$

$\frac{\cancel{6}^{3} x^{\cancel{3}^{1}}}{\cancel{2}x^{2}}$ $\frac{-\cancel{10}^{-5}\cancel{x^2}}{\cancel{2}\cancel{x^2}}$

Note that the exponent of the variable of the GCF corresponds to the lowest exponent in all terms.

3 Factoring "Simple" Trinomials

By the distributive property of mathematics, we can expand the expression $(ax+b)(cx+d)$.

$$ax \cdot cx + ax \cdot d + b \cdot cx + b \cdot d$$

$$(a \cdot c)x^2 + (a \cdot b \cdot c \cdot d)x + b \cdot d$$

When both $a = 1$ and $c = 1$, this produces:

$$(x+b)(x+d)$$

$$= x^2 + dx + bx + bd$$

$$= x^2 + (b+d)x + b \cdot d$$

ADD

MULT

There is a relationship between the values of the constants b and d, and the resulting expression.

A *simple quadratic trinomial* has the form $x^2 + bx + c$. We can use the previous relationship to factor it as two linear binomials.

For example, $x^2 + 11x + 10$ factors as:

ADD: 11

MULT: 10 → 1, 2, 5, 10

$(x+1)(x+10)$

$1 \times 10 = 10$, $1 + 10 = 11$

Using the signs of the constant c and the coefficient b can help us make informed decisions about the factors.

For example, $x^2 + 3x - 18$ factors as:

MULT: −18 (ONE POS, ONE NEG)

$(x+6)(x-3)$

ADD: 3

Some simple trinomials, such as $x^2 - 3x + 8$, cannot be factored.

8: 1, 2, 4, 8

$1 \times 8 = 8$, $1 + 8 = 9$

$2 \times 4 = 8$, $2 + 4 = 6$

4 Factoring By Grouping

Some expressions have terms which have the same common factors as others, but not all of them. In these situations, it may be possible to determine a common factor by *grouping* terms.

For example, the expression $\underbrace{x^3 + 3x^2}_{\text{COMMON FACTOR } x^2} \underbrace{- 6x - 18}_{\text{COMMON FACTOR } -6}$ factors as:

SIGN CHANGE!

$$x^2(x+3) - 6(x+3)$$

$$= (x+3)(x^2-6)$$

5 Factoring "Complex" Trinomials

While grouping may be used on its own, it is more likely to be used as part of the process of factoring *complex quadratic trinomials*.

A complex quadratic trinomial has the form $ax^2 + bx + c$ where $a \neq 0$ and $a \neq 1$. Unlike simple quadratic trinomials, we can not use the relationship between the constant and coefficient, since we must take into account the effect of the value of a.

Instead, we use a process to "decompose" the bx term into a state where we can factor by grouping.

$$ax^2 + bx + c$$

$$j + k = b$$

$$j \cdot k = a \cdot c$$

$$ax^2 + \underbrace{jx + kx}_{j+k=b} + c$$

FIND COMON FACTOR OF ax^2 AND jx

FIND COMON FACTOR OF kx AND c

$4(-6) = -24$

For example, the expression $4x^2 + 5x - 6$ factors as:

$8 + (-3) = 5$

$8(-3) = -24$

$\underbrace{4x^2 + 8x}\ \underbrace{-3x - 6}$

$4x(x+2) - 3(x+2)$

$(x+2)(4x-3)$

6 Special Cases

Factoring using the previous method can be tedious, but there are some cases where we can use shortcuts.

Consider the expression $4x^2 - 25$. This is not a trinomial, but we can write it as one as:

$$4x^2 + 0x - 25$$

We can then use the previous method to factor:

$$4x^2 + 10x - 10x - 25$$

$$2x(2x+5) - 5(2x+5)$$

$$(2x-5)(2x+5)$$

$\sqrt{4} = 2$

SAME "VALUES", DIFFERENT SIGNS ON CONSTANTS

$\sqrt{25} = 5$

Note the relationship between the constants and coefficients in the factored expression, and those of the non-factored expression.

This is an example of a *difference of squares*.

Now consider the expression $4x^2 - 20x + 25$. It factors as:

$$4x^2 - 10x - 10x + 25$$

$$2x(2x-5) - 5(2x-5)$$

$$(2x-5)(2x-5)$$

$$(2x-5)^2$$

$\sqrt{4} = 2$ $\sqrt{25} = 5$

Note the relationship between the constants and coefficients in the factored expression, and those of the non-factored expression.

This is an example of a *perfect square*.

7 Multi-Stage Factoring

Some expressions can be factored multiple times, using one or more of the previous techniques.

For example, consider the expression $5x^3 - 20x$. This factors as:

COMMON FACTOR FIRST

$$GCF = 5x$$

$$5x(\underbrace{x^2-4}_{\text{DIFF OF SQUARES}})$$

$$5x(x+2)(x-2)$$

When asked to factor an expression or equation, be sure to factor it fully.

8 Using Algebraic Substitution

In some situations, it may be easier to substitute a temporary variable to simplify an expression before factoring.

For example, it is not necessary to expand $9(x+5)^2 + 12(x+5) + 4$ before factoring it.

let $k = x + 5$

$\sqrt{9} = 3$ $\qquad$ $\sqrt{4} = 2$

$9k^2 + 12k + 4$

$3 \times 2 \times 2 = 12$

$(3k + 2)^2$

$[(3(x+5)+2)^2]$

$(3x + 15 + 2)^2$

$(3x + 17)^2$