MDM4U: Mathematics of Data Management

What Is the Likelihood That...?

Probability Basics

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A Number Game

Work with a partner. Each player has three cards, 1-3, from which one card is randomly drawn.

Let P be the product of the two numbers, and S the sum. Then:

- Player 1 gets a point if $P < S$.
- Player 2 gets a point if $P > S$.
- Neither player gets a point if $P = S$.

Replace the cards after each turn. Play the game 20 times. Who is the winner?

A Number Game

Who has the advantage in the number game?

Use a table to tabulate the results (sum, product). Player 1's wins are shown in red, Player 2's in blue.

	1	2	3
1	(2, 1)	(3, 2)	(4, 3)
2	(3, 2)	(4, 4)	(5, 6)
3	(4, 3)	(5, 6)	(6, 9)

Player 1 wins in five cases, whereas Player 2 wins in only 3. There is only one case where neither player gets a point.

Terminology

An *experiment* is a sequence of *trials* in which some result is observed.

An *outcome* is a result of an experiment.

The *sample space* is the set of all possible outcomes (i.e. the universal set).

An *event* is a subset of the sample space.

Terminology

Example

A six-sided die is rolled and a 5 is face up. Identify the experiment, outcome, and sample space.

The experiment was *rolling the die*.

The outcome was 5.

The sample space is *the numbers* 1-6, or $S = \{1, 2, 3, 4, 5, 6\}$.

Terminology

Different events can be associated with the sample space.

Let E be the event *rolling an even number*. Then $E = \{2, 4, 6\}$.

Let P be the event *rolling an even, prime number*. Then $P = \{2\}$.

Event P consists of only one outcome, and is called a *simple event*.

Terminology

Your Turn

A tertrahedral die has the numbers 1-4 marked on its four faces. One face always lands face down. Let the experiment be rolling the die, and the outcome the downward face.

List the elements of the sample space. $S = \{1, 2, 3, 4\}$

List the members of the event E, rolling an even square number. $E = \{4\}$

Describe the event $A = \{1, 3, 4\}$. Two is not rolled, etc.

Which of E or A is a simple event? E is a simple event.

Terminology

Probability is a number which indicates how likely an event is to occur.

There are three basic types of probability:

- Subjective – based on informed guesswork
- Empirical (or experimental) – based on direct observation and experimentation
- Theoretical – based on mathematical analysis

We deal mainly with the third type in this course, occasionally using empirical probability when appropriate.

Empirical Probability

The experimental probability, $P(E)$, of an event E is,

$$P(E) = \frac{\text{number of times } E \text{ has occurred}}{\text{number of trials in the experiment}}$$

Example

A six-sided die is tossed ten times, and a 4 is thrown twice. Let E be the event *a 4 is thrown*. What is the probability of throwing a 4?

E has occurred twice, out of ten experimental trials.

Thus, $P(E) = \frac{2}{10} = \frac{1}{5}$.

Theoretical Probability

The theoretical probability, $P(E)$, of an event E is,

$$P(E) = \frac{n(E)}{n(S)}$$

Example

What is the probability of throwing a 4 on a six-sided die?

$E = \{4\}$, so $n(E) = 1$.

$S = \{1, 2, 3, 4, 5, 6\}$, so $n(S) = 6$.

Thus, $P(E) = \frac{1}{6}$.

Empirical and Theoretical Probability

Consider a die with the number 5 on all of its six faces. What is the probability of rolling a 5?

Let A be the event *rolling a 5*. Then $n(A) = 1$, $n(S) = 1$ and $P(A) = \frac{1}{1} = 1$.

Using the same die, what is the probability of rolling a 4?

Let B be the event *rolling a 4*. Then $n(B) = 0$, $n(S) = 1$, and $P(B) = \frac{0}{1} = 0$.

The probability of an event occuring is a value between 0 and 1. An event that *never* occurs has a probability of 0, while an event that *always* occurs has a probability of 1.

Empirical and Theoretical Probability

In most cases, the empirical probability of an event will differ from the theoretical probability.

With only a few trials, it is very likely that the same outcome will occur multiple times, skewing the data.

For example, in two coin tosses, heads comes up twice. The empirical probability of tossing heads is 1, whereas the theoretical probability is $\frac{1}{2}$.

In general, as the number of trials increases, the value of the empirical probability approaches that of the theoretical probability.

Compliment of an Event

For any event E, its *compliment* $\overline{E}$ or E' is the set of all outcomes where E does not happen.

For example, let an experiment be rolling a standard die and E be the event *a prime number is rolled*.

Then $E = \{2, 3, 5\}$ and $\overline{E} = \{1, 4, 6\}$.

Since E and $\overline{E}$ together include all possible outcomes, the sum of their probabilities must be 1.

$P(E) + P(\overline{E}) = 1$ or $P(\overline{E}) = 1 - P(E)$.

Compliment of an Event

Example

A card is drawn from a standard deck. What is the probability that it is a club? That it is not a club?

Solution: Of the 52 cards in the deck, 13 are clubs. Let C be the event *a club is drawn*.

Therefore, the probability of drawing a club is $P(C) = \frac{13}{52} = \frac{1}{4}$.

The probability that the card is *not* a club is $P(\overline{C}) = 1 - \frac{1}{4} = \frac{3}{4}$.

Compliment of an Event

Your Turn

What is the probability that a card drawn randomly from a standard deck is neither a face card, nor a heart?

Let F be the event *drawing a face card* and H the event *drawing a heart*.

There are twelve face cards in the deck, and thirteen hearts. Three cards are both face cards *and* hearts: J♡, Q♡ and K♡. Therefore, the number of cards that are either a face card *or* a heart is $12 + 13 - 3 = 22$.

The probability of drawing either a face card, or a heart, is $P(F \text{ or } H) = \frac{22}{52} = \frac{11}{26}$.

Thus, the probability that a randomly drawn card is neither a face card, nor a heart, is $P(\overline{F \text{ or } H}) = 1 - \frac{11}{26} = \frac{15}{26}$.

Questions?