MDM4U: Mathematics of Data Management

Arranging Distinct Items

Permutations

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Recap

Example

In how many ways can six marbles be arranged in a straight line?

There are six choices for the leftmost marble, then five choices for the next, then four for the next...

By the FCP, there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$ ways.

Permutations

A *permutation* is an arrangement of items where order is important.

In the previous example, the six marbles were arranged in a specific order. Each arrangement is unique, even though the same six marbles are used for each arrangement.

Note that six items were arranged in 6! ways.

Permutations of All Items

Permutations of n Distinct Items

Given n distinct items, the number of permutations of all n items, denoted ${}_nP_n$ or $P(n, n)$, is $n!$

Proof: There are n ways in which to choose the first item, $n - 1$ ways in which to choose the second, and so forth, until there is only one choice for the last item.

According to the FCP,

$$\begin{aligned} {}_nP_n &= n(n-1)(n-2)\ldots(2)(1) \\ &= n! \end{aligned}$$

Permutations of All Items

Example

Determine the number of ways in which four elected students can fill the positions of President, Vice President, Treasurer and Secretary of school council.

Imagine arranging the four students in a line, such that the President is always at the left, followed to the right by the Vice President, Treasurer and Secretary, in that order.

Therefore, there are ${}_4P_4 = 4! = 24$ ways in which the four students can fill the positions.

In situations where positions carry different responsibilities, we are almost always talking about permutations.

Permutations of All Items

Your Turn

In how many ways can a dozen different donuts be distributed evenly among twelve children?

Imagine the twelve children standing in a line. Alternatively, think of each child's name as their "position."

Therefore, there are ${}_{12}P_{12} = 12! = 479\,001\,600$ ways to split the donuts.

The problem becomes more difficult if a child may receive *any* number of donuts, or if only *some* children receive donuts.

Permutations of Some Items

Sometimes we only want to arrange *some* items, taken from a collection.

For example, suppose an artist brings five paintings to an exhibition. She can hang only two paintings, next to each other, in the main gallery. In how many ways can she do this?

One approach might be to label the paintings A-E, and enumerate all 20 possible arrangements.

AB, AC, AD, AE, BA, BC, BD, BE, CA, CB,
CD, CE, DA, DB, DC, DE, EA, EB, EC, ED

Permutations of Some Items

A better approach is to use the FCP, using 5 choices for the first painting and 4 for the second, giving $5 \times 4 = 20$ ways.

Note that

$$\begin{aligned} 5 \times 4 &= 5 \times 4 \times \frac{3 \times 2 \times 1}{3 \times 2 \times 1} \\ &= \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} \\ &= \frac{5!}{3!} \\ &= \frac{5!}{(5-2)!} \end{aligned}$$

Permutations of Some Items

Permutations of r Items, Taken From n Distinct Items

Given n distinct items, the number of permutations of r items, denoted ${}_nP_r$ or $P(n,r)$, is $\dfrac{n!}{(n-r)!}$

Proof: There are n ways to choose the first item, $n-1$ ways to choose the second item, and so forth, until there are $n-r+1$ choices for the rth item.

For example, for ${}_5P_3$ we have 5 choices for the first item, 4 choices for the second, and 3 choices for the third. $n-r+1 = 5-3+1 = 3$.

continued...

Permutations of Some Items

By the FCP,

$$\begin{aligned} {}_nP_r &= n(n-1)(n-2)\dots(n-r+1) \\ &= n(n-1)(n-2)\dots(n-r+1) \times \frac{(n-r)!}{(n-r)!} \\ &= \frac{n(n-1)(n-2)\dots(n-r+1)(n-r)(n-r-1)\dots(2)(1)}{(n-r)!} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Permutations of Some Items

Example

Determine the number of ways in which three students can stand in a line, taken from a group of eight students.

$$\begin{aligned} {}_8P_3 &= \frac{8!}{(8-3)!} \\ &= \frac{8!}{5!} \\ &= 8 \times 7 \times 6 \\ &= 336 \end{aligned}$$

Three of eight students can line up in 336 ways.

Permutations of Some Items

Your Turn

Six cards are dealt from a standard deck, one after the other, and arranged in a line. Determine the number of ways in which this can occur.

$$\begin{aligned} {}_{52}P_6 &= \frac{52!}{(52-6)!} \\ &= \frac{52!}{46!} \\ &= 52 \times 51 \times 50 \times 49 \times 48 \times 47 \\ &= 14\,658\,134\,400 \end{aligned}$$

There are over 14 billion ways to arrange the six cards!

Permutations with Conditions

In some cases, we need to arrange items according to certain conditions. For example, an item may need to be in a certain position, or two items may need to be adjacent.

Example

Mr. Garvin and nine of his Data Management students are seated in a row in the auditorium. If Mr. Garvin always occupies the aisle seat, determine the number of ways in which the students can be seated.

Mr. Garvin's position is fixed, so only nine items (the students) are arranged.

This can be done in $9! = 362\,880$ ways.

Permutations with Conditions

Example

In how many ways can Mr. Garvin's nine students be seated, if Sophie and Claire insist on sitting next to each other?

Since Sophie and Claire sit next to each other, treat them as a single item. There are now eight items to arrange, which can be done in $8! = 40\,320$ ways.

However, Sophie and Claire can arrange themselves in two ways (SC or CS), so we must account for this by multiplying the previous answer by two.

Thus, the nine students can be seated in $2 \times 8! = 80\,640$ ways.

Permutations with Conditions

Your Turn

In how many ways can Mr. Garvin's nine students be seated, if Devin always sits between Holly and Michael?

Group the three students as one item. Now there are seven items to arrange, which can be done in $7! = 5\,040$ ways.

The three students can sit as HDM or MDH, so multiply the answer by two to get $2 \times 5\,040 = 10\,080$ possible arrangements.

Indirect Method of Counting

Example

In how many ways can Mr. Garvin's nine students be seated if Vincent and Ivan refuse to sit next to each other?

Recall that some cases are easier to analyse using an *indirect method*: subtract the number of unacceptable possibilities from the total number of possibilities.

With no restrictions, there are $9! = 362\,880$ ways of seating nine students.

If Vincent and Ivan *do* sit together, there are $2 \times 8! = 80\,640$ ways of seating the students.

Thus, there must be $9! - 2 \times 8! = 282\,240$ seating arrangements in which Vincent and Ivan are *not* together.

Indirect Method of Counting

Your Turn

In how many ways can the letters of the word THINK be arranged, such that the T and the H are not adjacent?

There are $5! = 120$ ways of arranging the five letters.

There are $2 \times 4! = 48$ ways of arranging the five letters, such that the T and the H *are* adjacent.

Therefore, there are $5! - 2 \times 4! = 72$ ways to arrange the five letters, such that the T and the H are *not* adjacent.

Questions?