MDM4U: Mathematics of Data Management

Subsets and Selections

Other Counting Techniques

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Sums of Money

How many different sums of money can be made using two dimes, one quarter and four pennies?

1¢, 2¢, 3¢, 4¢, 10¢, 11¢, 12¢, 13¢, 14¢, 20¢, 21¢, 22¢, 23¢, 24¢, 25¢, 26¢, 27¢, 28¢, 29¢, 35¢, 36¢, 37¢, 38¢, 39¢, 45¢, 46¢, 47¢, 48¢, 49¢

There are 29 different sums of money that can be formed using some combination of the seven coins.

Enumerating them is a tedious process. Is there a faster way to *count* the number of possibilities?

Sums of Money

To count the possibilities, consider the number of ways each "group" of coins could be treated.

For the dimes, a sum of money could include both dimes, one dime, or no dimes at all.

Therefore, there are three ways of treating the dimes.

Similarly there are two ways of treating the quarter (it is included or it is not), and five ways of treating the pennies (four, three, two, one or none are selected).

According to the FCP, this gives $3 \times 2 \times 5 = 30$ sums of money.

Sums of Money

So why did we count only 29 possibilities?

When treating each group of coins, we allowed the option of selecting no coin from that group. Therefore, it is possible that *no coins were selected at all*.

Since a sum of money must include at least one coin, we must eliminate this single possibility.

Therefore, $3 \times 2 \times 5 - 1 = 29$ valid sums.

Selection Problems

Selections Involving Some Identical Items

Given a items of one type, b of another, c of another, and so forth, the total number of selections that can be made is:

- $(a+1)(b+1)(c+1)\dots$ if the empty set is included
- $(a+1)(b+1)(c+1)\dots - 1$ if the empty set is not included

Selection Problems

Example

A bag contains four red marbles, three green marbles and one yellow marble. Holly reaches into the bag and randomly draws one or more marbles. How many different selections are there?

Use the formula for the number of selections without the empty set, since Holly draws at least one marble.

Therefore, there are $(4+1)(3+1)(1+1) - 1 = 5 \times 4 \times 2 - 1 = 39$ different selections.

Selection Problems

Example

The English department needs five copies of *Romeo & Juliet*, three copies of *Slaughterhouse Five* and eight dictionaries. Due to budget cuts, they may not be able to order any or all of the books this semester. How many different ordering options are available for the English department?

It is possible that the English department has *no* money to spend on books at all, so the null set is a valid option here.

Therefore, there are $(5+1)(3+1)(8+1) = 6 \times 4 \times 9 = 216$ options.

Counting Subsets

Number of Subsets of a Set

A set with n elements has 2^n subsets.

Proof: Use the previous forumla, $(a+1)(b+1)(c+1)\ldots$. Since each element in a set is unique, $a = b = c = \ldots = 1$.

Substituting into the forumla yields
$(1+1)(1+1)(1+1)\ldots = \underbrace{2 \times 2 \times 2 \times \ldots \times 2}_{n \text{ times}} = 2^n$

Counting Subsets

Example

List all subsets of the set $S = \{a, b, c\}$.

There are three elements in S, so there must be $2^3 = 8$ subsets.

$\{a, b, c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a\}$, $\{b\}$, $\{c\}$

Where is the eighth subset?

The null set, $\emptyset$, is *always* a subset!

Counting Subsets

Example

Chris arrives late to the cafeteria for lunch, and there are only five items (hot dog, fries, salad, pop, cookie) left for sale. How many different purchases can he make?

If each item is considered an element of a set, then there are five elements: $S = \{h, f, s, p, c\}$.

Therefore, there are $2^5 = 32$ possible purchases he can make.

BUT, this calculation includes the null set (Chris purchases nothing). This is not really a purchase at all, so we must eliminate this possibility.

Therefore, there are $2^5 - 1 = 31$ possible purchases he can make.

Problem Solving with Combinations

When dealing with combinations, it is important to identify if the null set is a valid option or not.

If it is invalid, then it must be subtracted from the possible outcomes.

It is also important to *think* about cases where different options may produce the same outcome.

Sums of Money (Revisited)

Example

How many different sums of money can be made using one \$5 bill, two \$10 bills, and one \$20 bill?

Using the formula for selections of identical items, there are $(1+1)(2+1)(1+1) = 2 \times 3 \times 2 - 1 = 11$ different sums.

A sum of \$20, however, can be made two ways: using the \$20 bill, or the two \$10s.

Enumerating the possibilities reveals that there are only 9 possible sums: \$5, \$10, \$15, \$20, \$25, \$30, \$35, \$40 and \$45.

The formulas developed in this chapter can only be used when there is no possibility of the same result being produced in two different ways.

Questions?