MDM4U: Mathematics of Data Management

How Long Until You Succeed?

Geometric Probability Distributions

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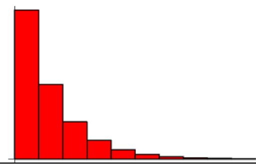

Geometric Probability Distributions

Recap

A simple game is played, where the player must roll a 1 or a 6 on a die to win. Determine the probability distribution for rolling a 1 or a 6 on the first, second or third roll.

A success is rolling a 1 or a 6. The probability of success is $\frac{1}{3}$.

Rolls	Probability
1	$\frac{1}{3}$
2	$\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$
3	$\frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{27}$
$\vdots$	$\vdots$

Geometric Probability Distributions

The previous example consisted of a (potentially infinite) sequence of Bernoulli trials.

Instead of the *number of successes*, we are interested in the *waiting time* until the first success occurs.

Rolling a 1 or a 6 immediately corresponds to a waiting time of zero rolls.

Rolling a 1 or a 6 on the third roll corresponds to a waiting time of two rolls.

Geometric Probability Distributions

Probability in a Geometric Probability Distribution

The probability that the first successful outcome occurs after x failures, in a sequence of Bernoulli trials with probability of success p and probability of failure q, is
$P(x) = q^x p$

This should be intuitive, since all failures must occur before the first success.

According to the Product Rule for independent events, each failure adds a factor of q to the probability.

Therefore, x failures will occur with a total probability of q^x, followed by a success with probability p. This gives the equation above.

Geometric Probability Distributions

Example

A fair coin is tossed repeatedly until it comes up heads. What is the probability that the first head occurs on the fifth toss?

If a head occurs on the fifth toss, there must be four tails that precede it. So $x = 4$, $p = \frac{1}{2}$ and $q = \frac{1}{2}$.

$P(4) = \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right) = \frac{1}{32}$.

Geometric Probability Distributions

Your Turn

Juliana randomly presses letter keys on Mr. Garvin's laptop one at a time. What is the probability that the third press will be the first vowel?

If a vowel occurs on the third press, there must be two consonants that precede it. So $x = 2$, $p = \frac{5}{26}$ and $q = 1 - \frac{5}{26} = \frac{21}{26}$.

$P(2) = \left(\frac{21}{26}\right)^2 \left(\frac{5}{26}\right) = \frac{2205}{17576}$.

Geometric Probability Distributions

Example

On average, a soccer goalie saves 9 out of of every 12 shots. What is the probability that the first goal scored occurs on the third shot or later?

If a goal is scored on the third shot, there must be two saves that precede it. This would be $P(2)$.

If the goal is scored in the fourth shot, there must be three saves that precede it. This would be $P(3)$.

The only instances that we are *not* concerned about are $P(0)$ (a goal is scored immediately) and $P(1)$ (one save is made).

Geometric Probability Distributions

What constitutes a success? In this case, we are looking for the first goal scored.

A *success* is when the goalie *fails* to save the shot.

So, $x \geq 2$, $q = \frac{9}{12} = \frac{3}{4}$ and $p = 1 - \frac{3}{4} = \frac{1}{4}$.

$$\begin{aligned} P(x \geq 2) &= 1 - P(0) - P(1) \\ &= 1 - \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right) - \left(\frac{3}{4}\right)^1 \left(\frac{1}{4}\right) \\ &= 1 - \frac{1}{4} - \frac{3}{16} \\ &= \frac{9}{16} \end{aligned}$$

Expected Value

Expected Value in a Geometric Probability Distribution

For a geometric probability distribution of Bernoulli trials, each with probability of success p and probability of failure q, the expected value is
$E(X) = \frac{q}{p}$.

Since calculus is required to show that this infinite series converges toward $\frac{q}{p}$, the proof is omitted. But feel free to do research.

Expected Value

Example

What is the expected number of rolls before doubles is thrown on two dice?

Let D be the event where doubles are thrown. Then there are six ways to throw doubles, for a probability of $P(D) = \frac{6}{36} = \frac{1}{6}$.

The probability of *not* throwing doubles is $P(\overline{D}) = 1 - \frac{1}{6} = \frac{5}{6}$.

So the expected number of rolls before doubles is thrown is $E(X) = \frac{5}{6} \div \frac{1}{6} = 5$.

Expected Value

Your Turn

A recent poll indicates that the Green Party has 9% popular support. How many people should a reporter expect to interview before he finds a Green Party supporter?

Let G be the event that a Green Party supporter is found. Then $P(G) = \frac{9}{100}$.

The probability that a Green Party supporter is *not* found is $P(\overline{G}) = 1 - \frac{9}{100} = \frac{91}{100}$.

The expected waiting time is $E(X) = \frac{91}{100} \div \frac{9}{100} = \frac{91}{9}$ or just more than 10 people.

Therefore, the reporter should expect to interview 11 people.

Expected Value

Example

Gabriella has 8 marbles in a bag: 4 each of red and blue. For each game of marbles she plays, she randomly draws one marble out of the bag, plays, then returns it to the bag. How long should she expect to wait until she has played with both colours?

On the first draw, Gabriella draws a red or blue marble. She plays 1 round with this marble.

There is a $\frac{4}{8} = \frac{1}{2}$ probability that a subsequent marble will be a different colour. Therefore, the expected wait time until a second colour is drawn is $E(x) = \frac{1}{2} \div \frac{1}{2} = 1$.

Therefore, the expected wait time until she plays with both colours is $1 + 1 = 2$ games.

Questions?