MDM4U: Mathematics of Data Management

The Big Bang Theory

Factorial Notation

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Calculate

$$4 \times 3 \times 2 \times 1 = 24$$

$$5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

Factorial Notation

Factorials

The factorial of a whole number n is the product of all positive integers less than or equal to n.

Factorials use the notation $n!$.

Most people read $n!$ as "n factorial," while others read $n!$ as "n bang."

Factorial Notation

Factorials (Mathematical Definition)

For any whole number n, $n! = n(n-1)(n-2)\dots(3)(2)(1)$.

Factorials get very large, very fast.

$$12! = 12 \times 11 \times \dots \times 2 \times 1 = 479\,001\,600$$

$$13! = 13 \times 12! = 6\,227\,020\,800$$

$$14! = 14 \times 13! = 87\,178\,291\,200$$

Evaluating Factorials

Evaluate 75! using your calculator. What is the result?

Most scientific calculators cannot handle a number this large.

For example, a TI-83 graphing calculator can only handle up to 69!, which has a value of

171,122,452,428,141,311,372,468,338,881,272, 839,092,270,544,893,520,369,393,648,040,923, 257,279,754,140,647,424,000,000,000,000,000

which is 99 digits long!

This does not mean that large factorials can not be used in calculations.

Evaluating Factorials

Consider the expression $\frac{75!}{71!}$. How can we evaluate this?

Note that $75! = 75 \times 74 \times 73 \times 72 \times 71 \times \dots \times 2 \times 1$

Note that $71! = 71 \times \dots \times 2 \times 1$

Therefore,

$$\begin{aligned}\frac{75!}{71!} &= \frac{75 \times 74 \times 73 \times 72 \times 71 \times \dots \times 2 \times 1}{71 \times \dots \times 2 \times 1} \\ &= 75 \times 74 \times 73 \times 72 \\ &= 29\,170\,800\end{aligned}$$

In this way, we can simplify expressions involving large factorials that a calculator might not be able to handle.

Evaluating Factorials

Your Turn

Simplify $\frac{80!}{77!}$.

$$\begin{aligned}\frac{80!}{77!} &= \frac{80 \times 79 \times 78 \times 77 \times \ldots \times 2 \times 1}{77 \times \ldots \times 2 \times 1}\\ &= 80 \times 79 \times 78\\ &= 492\,960\end{aligned}$$

Simplifying Factorials

Your Turn

Simplify $\frac{24!}{21!12!}$.

$$\begin{aligned}\frac{24!}{21!12!} &= \frac{24 \times 23 \times 22 \times 21 \times \ldots \times 2 \times 1}{21 \times \ldots \times 2 \times 1 \times 12!}\\ &= \frac{24 \times 23 \times 22}{12 \times 11 \times \ldots \times 2 \times 1}\\ &= \frac{(6 \times 4) \times 23 \times (2 \times 11)}{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}\\ &= \frac{23}{12 \times 10 \times 9 \times 8 \times 7 \times 5 \times 3 \times 1}\\ &= \frac{23}{907\,200}\end{aligned}$$

Simplifying Factorials

Factorials can also be simplified more generally.

Example

Simplify $n(n-1)!$.

By definition, $(n-1)! = (n-1)(n-2)\ldots(2)(1)$.

$$\begin{aligned}n(n-1)! &= (n)(n-1)(n-2)\ldots(2)(1)\\ &= n!\end{aligned}$$

Simplifying Factorials

Example

Simplify $\frac{(n+1)!}{n!}$.

$$\begin{aligned}\frac{(n+1)!}{n!} &= \frac{(n+1)(n)(n-1)\ldots(2)(1)}{(n)(n-1)\ldots(2)(1)}\\ &= n+1\end{aligned}$$

Simplifying Factorials

Your Turn

Simplify $\frac{(n+1)!}{(n-1)!}$.

$$\begin{aligned}\frac{(n+1)!}{(n-1)!} &= \frac{(n+1)(n)(n-1)\ldots(2)(1)}{(n-1)\ldots(2)(1)}\\ &= (n+1)(n)\\ &= n^2+n\end{aligned}$$

Solving Factorial Expressions

Expressions involving factorials can also be *solved*.

Example

Solve $\frac{(n+2)!}{n!} = 12$.

$$\begin{aligned}\frac{(n+2)(n+1)(n)\ldots(2)(1)}{(n)\ldots(2)(1)} &= 12\\ (n+2)(n+1) &= 12\\ n^2+3n+2 &= 12\\ n^2+3n-10 &= 0\\ (n+5)(n-2) &= 0\\ n &= \{-5, 2\}\end{aligned}$$

Solving Factorial Expressions

BUT...

Recall that $n!$ must be a whole number.

If $n = -5$, then $\frac{(n+2)!}{n!} = \frac{(-5+2)!}{n!} = \frac{(-3)!}{(-5)!}$.

Both $(-5)!$ and $(-3)!$ are not defined, so we must reject $n = -5$ as a solution.

Thus, the *only* solution is $n = 2$.

COUNTING PRINCIPLES AND PERMUTATIONS

Questions?