MDM4U: Mathematics of Data Management

Depending On Others

Dependent and Independent Events

J. Garvin

Independent Events

Independent events are those that do not influence the other.

For example, the toss of a coin does not influence the roll of a die, and the rain outside does not influence the value of a card drawn from a Euchre deck.

To calculate the probability of both events occurring, independent of each other, we can use the Product Rule for independent events.

Independent Events

Product Rule for Independent Events

If events A and B are independent events, then the probability of both A and B occurring is $P(A \cap B) = P(A) \times P(B)$

Proof:

Recall that $P(A) = \frac{n(A)}{n(S_A)}$ and $P(B) = \frac{n(B)}{n(S_B)}$.

Let S_{AB} be the combined sample space for events A and B. Then $P(A \cap B) = \frac{n(A \cap B)}{n(S_{AB})}$.

Independent Events

Since A and B are independent, use FCP to get $n(A \cap B) = n(A) \times n(B)$.

Since the sample spaces are also independent, use FCP to get $n(S_{AB}) = n(S_A) \times n(S_B)$.

Therefore, substituting into the equation gives...

$$\begin{aligned} P(A \cap B) &= \frac{n(A) \times n(B)}{n(S_A) \times n(S_B)} \\ &= \frac{n(A)}{n(S_A)} \times \frac{n(B)}{n(S_B)} \\ &= P(A) \times P(B) \end{aligned}$$

Independent Events

To illustrate the Product Rule for independent events, consider the act of throwing two dice. What is the probability that a pair of 4s is tossed?

Let D_1 be the event *a 4 is tossed on die 1* and D_2 the event *a 4 is tossed on die 2*. The first toss in no way affects the second toss, so the two events are independent.

Since there is only one way to toss a 4 on a die, $n(D_1) = n(D_2) = 1$.

The sample spaces are identical, and are independent of each other – the first die's outcome does not affect the sample space for the second die. Therefore, $n(S_{D1}) = n(S_{D2}) = 6$.

According to the product rule, the probability of throwing a 4 then throwing another 4 is $P(D_1 \cap D_2) = P(D_1) \times P(D_2) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

Independent Events

This is consistent with our knowledge of dice. There is exactly 1 outcome of 36 possible outcomes where two 4s are tossed.

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Independent Events

Example

A game consists of flipping a fair coin, followed by drawing a card from a standard deck. What is the probability that a player tosses heads and draws a face card?

Let H be the event *tossing heads* and F the event *drawing a face card*.

Then $P(H) = \frac{1}{2}$ and $P(F) = \frac{12}{52} = \frac{3}{13}$.

Therefore, $P(H \cap F) = \frac{1}{2} \times \frac{3}{13} = \frac{3}{26}$.

Independent Events

Again, we can verify the probability by locating the relevant outcomes within the sample space. Note that $\frac{12}{108} = \frac{3}{26}$.

	1	2	3	4	5	6	7	8	9	10	J	Q	K
H	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠
H	♡	♡	♡	♡	♡	♡	♡	♡	♡	♡	♡	♡	♡
H	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣
H	♢	♢	♢	♢	♢	♢	♢	♢	♢	♢	♢	♢	♢
T	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠
T	♡	♡	♡	♡	♡	♡	♡	♡	♡	♡	♡	♡	♡
T	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣
T	♢	♢	♢	♢	♢	♢	♢	♢	♢	♢	♢	♢	♢

Independent Events

Example

A student estimates that his probability of passing Data Management is $\frac{4}{5}$, while his probability of passing English is $\frac{9}{10}$. Determine the probability that he will pass both courses.

Let D be the event *the student passes Data Management*, and E the event *the student passes English*.

$$\begin{aligned} P(D \cap E) &= P(D) \times P(E) \\ &= \frac{4}{5} \times \frac{9}{10} \\ &= \frac{36}{50} \\ &= \frac{18}{25} \end{aligned}$$

Independent Events

Your Turn

Using the same probabilities as before, determine the probability that he will *fail* both courses.

Let $\overline{D}$ be the event *the student fails Data Management*, and $\overline{E}$ the event *the student fails English*.

$$\begin{aligned} P(\overline{D} \cap \overline{E}) &= P(\overline{D}) \times P(\overline{E}) \\ &= \left(1 - \tfrac{4}{5}\right) \times \left(1 - \tfrac{9}{10}\right) \\ &= \frac{1}{5} \times \frac{1}{10} \\ &= \frac{1}{50} \end{aligned}$$

Note that the probability of failing both courses is not $1 - \frac{18}{25} = \frac{7}{25}$.

Dependent Events

Two events may not be independent of each other.

Consider drawing a card from a standard deck. What is the probability of drawing the Jack of Spades...

...from a standard deck? $P(J♠) = \frac{1}{52}$

...if you know the drawn card is black? $P(J♠ \text{ if } B) = \frac{1}{26}$

...if you know the drawn card is a face card? $P(J♠ \text{ if } F) = \frac{1}{12}$

...if the drawn card is red? $P(J♠ \text{ if } R) = 0$

Note that in each scenario, we are given additional information about the card that reduces the size of our sample space S.

Dependent Events

Product Rule for Dependent Events

If events A and B are dependent events, then the probability of B occurring, given that A has occurred, is $P(A \cap B) = P(A) \times P(B|A)$

The notation $P(B|A)$ is read "the probability of B, given that A has occurred" or "the probability of B if A."

It is almost identical to the Product Rule for independent events, but we must first determine the *conditional probability* $P(B|A)$.

Dependent Events

To see where the Product Rule for dependent events comes from, it is useful to rearrange the formula into its alternative representation.

$P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

Once A is known, the possible outcomes for B are restricted to those in $P(A \cap B)$. A has already occurred, so the only outcomes that involve B must involve A as well.

Since A has already occurred, the sample space has been reduced from S to A. Thus...

Dependent Events

$$\begin{aligned} P(B|A) &= \frac{n(A \cap B)}{n(A)} \\ &= \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(A)}{n(S)}} \\ &= \frac{P(A \cap B)}{P(A)} \end{aligned}$$

Dependent Events

Example

Determine the probability of dealing two Jacks, one after the other, from a standard deck.

There is a $\frac{4}{52} = \frac{1}{13}$ probability of dealing a Jack from the deck as the first card.

Since this card is not replaced, there is a $\frac{3}{51} = \frac{1}{17}$ probability of dealing one of the other three Jacks from the remaining 51 cards.

Therefore, the probability of dealing two Jacks back-to-back is $\frac{1}{13} \times \frac{1}{17} = \frac{1}{221}$.

Dependent Events

Example

Determine the probability of rolling a sum greater than 7 with two dice, if the first die rolled is a 3.

Let V be the event *a sum greater than seven is rolled*, and T the event *the first die is a three*.

Then the probability of rolling a sum greater than seven *and* the first die being a 3 is $P(V \cap T) = \frac{2}{36} = \frac{1}{18}$.

The probability of rolling a three is $P(T) = \frac{1}{6}$.

$$\begin{aligned} P(V|T) &= \frac{\frac{1}{18}}{\frac{1}{6}} \\ &= \frac{1}{3} \end{aligned}$$

Dependent Events

Your Turn

The probability that a student will attend Trent University is $\frac{1}{5}$. If she goes to Trent, the probability that her best friend will follow her is $\frac{3}{4}$. What is the probability that both students will go to Trent?

Let T be the event *the student goes to Trent*, and F the event the friend goes to Trent.

$P(T) = \frac{1}{5}$ and $P(F|T) = \frac{3}{4}$.

$$\begin{aligned} P(T \cap F) &= \frac{1}{5} \times \frac{3}{4} \\ &= \frac{3}{20} \end{aligned}$$

Questions?