MDM4U: Mathematics of Data Management

Counting Basics
Counting Principles

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Counting Principles

On a basic level, this course deals with *counting*. All probability concepts are based on the number of ways in which an event can occur.

One way to count things is to *enumerate* them, by listing all possible options.

Example

Given a fair coin, determine the number of ways in which tails is tossed exactly twice in three tosses.

There are eight possible outcomes, three of which contain exactly two tails.

$$HHH, HHT, HTH, THH, HTT, TTH, THT, TTT$$

Tree Diagrams

Another way to enumerate possible outcomes is to use a tree diagram.

Each branch of the tree diagram represents a "path" that can be followed to reach a specific outcome.

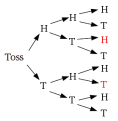
To enumerate the outcomes that satisfy a certain criterion, count the number of paths that lead to the desired result.

Tree Diagrams

Example

Given three tosses of a fair coin, determine the number of ways in the same face is *not* tossed in consecutive tosses.

There are eight possible paths in the tree diagram, two of which (HTH and THT) alternate heads and tails.

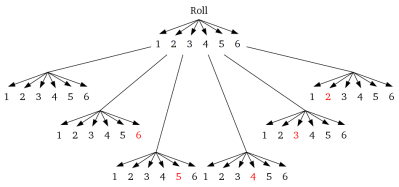


Tree Diagrams

Your Turn

Use a tree diagram to determine the number of ways in which a sum of 8 can be rolled on two dice.

There are five ways in which a sum of 8 can be rolled.



Alternate Solution

Another way to determine the solution is to use a 6×6 table for the two rolls.

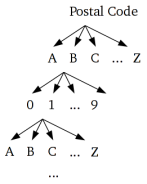
Table

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

This is a useful method for solving dice problems, and will come up throughout the course.

Tree Diagrams

Tree diagrams and tables are useful, but can get big quickly. For example, in how many ways can a Canadian postal code be made?



The diagram is too big to deal with reasonably. We need another method.

Fundamental Counting Principle (FCP)

To determine the number of ways that one action can be performed after another, use the Fundamental Counting Principle.

Fundamental Counting Principle

If one action can be performed in n ways, then another in m ways, then both actions can be performed, in order, in $n \times m$ ways.

The key word here is "then."

The FCP can be extended to any number of actions, one after the other.

Fundamental Counting Principle (FCP)

Example

In how many ways can a Canadian postal code be made? A postal code has the format A9A 9A9, where A is any letter and 9 is any number.

Think of this problem as selecting a letter, *then* selecting a number, *then* a letter, and so on.

There are 26 possible letters, and 10 possible numbers, for each position.

According to the FCP, the total number of postal codes is $26 \times 10 \times 26 \times 10 \times 26 \times 10 = 17\,576\,000$.

Fundamental Counting Principle (FCP)

Your Turn

A cafeteria offers lunch specials consisting of one item from each category.

Entree	**Beverage**	**Dessert**
Hamburger	Soft Drink	Ice Cream
Sandwich	Milk	Fruit Cup
Wrap	Juice	
Pasta		
Chicken Salad		

Determine the number of possible lunch specials.

Using the FCP, there are $5 \times 3 \times 2 = 30$ lunch specials.

Fundamental Counting Principle (FCP)

Sometimes, problems arise in which certain *restrictions* must be met.

Example

Determine the number of two-digit positive integers that are even.

A two digit positive integer cannot begin with a leading zero, so there are 9 possibilities for the first digit (1-9).

An even number is divisible by 2, so there are five possibilities for the second digit (0, 2, 4, 6, 8).

According the the FCP, there are $9 \times 5 = 45$ even two-digit positive integers.

Rule of Sum (RoS)

Recall the cafeteria menu from earlier.

Entree	**Beverage**	**Dessert**
Hamburger	Soft Drink	Ice Cream
Sandwich	Milk	Fruit Cup
Wrap	Juice	
Pasta		
Chicken Salad		

What if you only wanted one item from the cafeteria's menu. In how many ways can you select a single item?

There are five entrees, three beverages, and two desserts from which to choose. Therefore, there are $5 + 3 + 2 = 10$ ways of selecting a single item.

Rule of Sum (RoS)

The FCP cannot be used here, because one action does not occur after another. We are selecting one entree, *or* one beverage, *or* one dessert.

To determine the number of ways in which either one of two actions can be performed, use the Rule of Sum.

Rule of Sum

If one action can be performed in n ways, and another in m ways, and both actions cannot be performed together, then either action can be performed in $n + m$ ways.

The key word here is "or." The RoS can handle any number of actions, just like the FCP.

Rule of Sum (RoS)

Example

Determine the number of ways of drawing a red face card or a spade from a standard deck of 52 cards.

There are 6 red face cards (J♡, J♢, Q♡, Q♢, K♡, K♢) and 13 spades. Thus, according to the RoS, there are $6 + 13 = 19$ ways of drawing a red face card or a spade.

Rule of Sum (RoS)

Your Turn

In how many ways can a sum of either four or seven be rolled using a standard pair of dice?

There are three ways to roll a four (1-3, 3-1, 2-2) and six ways to roll a seven (1-6, 2-5, 3-4, 4-3, 5-2, 6-1).

Using the RoS, there are $3 + 6 = 9$ ways in which either a four or a seven can be rolled.

We can verify this using the 6×6 grid method.

Rule of Sum (RoS)

Example (Verification)

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Rule of Sum (RoS)

Example

In how many ways can a queen or a diamond be drawn from a standard deck?

There are 4 queens and 13 diamonds in the deck.

However...

...The queen of diamonds is counted twice. We cannot directly use the RoS to calculate the answer. But we *can* solve it!

Compensate for overcounting by subtracting the queen of diamonds. This gives $4 + 13 - 1 = 16$ ways.

We will talk about this later in the course when we cover sets.

Questions?