MDM4U: Mathematics of Data Management

Selecting a Dozen Donuts

Combinations with Repetition

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Selecting a Dozen Donuts

In how many ways can we select one dozen donuts from a large supply of chocolate, jelly and maple donuts?

Some possible arrangements include:

- CCCCJJJJMMMM
- CCJJJJJMMMMM
- CCCCCCCCCCCC (my favourite)

Selecting a Dozen Donuts

In this case, order does not matter, since CCCCJJJJMMMM is the same as MMMMJJJJCCCC. So we need to use combinations rather than permutations.

This time, however, we are allowed repetition, since the supply of donuts is very large (or possibly infinite).

This is different from previous questions such as forming committees or dealing poker hands, where selecting a person or object reduces the sample space.

To answer the donut question, let's first look at a simpler example.

A Simpler Question

In how many ways can we select three donuts from a large supply of chocolate and maple donuts?

There are four possible orders: CCC, CCM, CMM, MMM.

Another way of expressing the orders is to use a "divider" to separate the types of donuts. This gives us CCC/, CC/M, C/MM and /MMM.

Notice that if we always place the chocolate donuts on the left side of the divider, and the maple donuts on the right side, then there is no need to specify the type of donut explicitly.

This gives us XXX/, XX/X, X/XX and /XXX as representations for the four orders.

Note that there are three Xs and one / being arranged.

A Simpler Question

In how many ways can we select three donuts from a large supply of chocolate, jelly and maple donuts?

There are 10 possible orders:

Order	Rep.	Order	Rep.
CCC	XXX//	CMM	X//XX
CCJ	XX/X/	JJJ	/XXX/
CCM	XX//X	JJM	/XX/X
CJJ	X/XX/	JMM	/X/XX
CJM	X/X/X	MMM	//XXX

Here, chocolates always appear first, followed by jelly, followed by maple.

Note that there are three Xs and two /s being arranged.

Selecting Items With Repetition Allowed

If we concern ourselves with just the representations, the problem boils down to arranging two things: the r donuts selected, and the $n-1$ dividers separating the n types.

The r donuts are represented by the number of Xs, and the $n-1$ dividers are represented by the number of /s.

There is a total of $r+n-1$ things being arranged, with repetition.

From a *permutations* perspective, we know that this can be done in $\frac{(r+n-1)!}{r!(n-1)!}$ ways.

From a *combinations* perspective, note that this is the definition of ${}_{n+r-1}C_r$.

Selecting Items With Repetition Allowed

Generalized Combinations

The number of ways to select r items from n different types, with repetition allowed, is given by ${}_{n+r-1}C_r$.

In this case, it does not matter if r is greater than n.

For example, we can select 20 donuts from 3 types, or 3 donuts from 20 types.

Selecting Items With Repetition Allowed

Example

Confirm that there are 10 ways to select three donuts from a large supply of chocolate, jelly and maple donuts.

We are selecting 3 donuts from 3 types, which can be done in ${}_{3+3-1}C_3 = {}_5C_3 = 10$ ways.

Selecting a Dozen Donuts Revisited

Example

Returning to the original question: in how many ways can we select one dozen donuts from a large supply of chocolate, jelly and maple donuts?

We are selecting 12 donuts from 3 types, which can be done in ${}_{3+12-1}C_{12} = {}_{14}C_{12} = 91$ ways.

Example

What if the store offered 20 varieties of donuts instead?

With the additional types, the number of orderes increases dramatically to ${}_{20+12-1}C_{12} = {}_{31}C_{12} = 141\,120\,525$ ways.

Selecting a Dozen Donuts Revisited

Example

In how many ways can one dozen donuts be selected if the store offers 8 varieties of donuts, and *exactly* two chocolate donuts are selected?

When restrictions are in place, deal with the restrictions first and then calculate the rest.

In this case, place the two chocolate donuts in the box. Now we must select the 10 remaining donuts from the 7 non-chocolate types.

This can be done in ${}_{10+7-1}C_{10} = {}_{16}C_{10} = 8\,008$ ways.

Selecting a Dozen Donuts Revisited

Example

In how many ways can one dozen donuts be selected if the store offers 5 varieties of donuts, and *at least three* jelly donuts must be selected?

We place 3 jelly donuts in the box, then select the remaining 9 donuts from the 5 types.

This can be done in ${}_{5+9-1}C_9 = {}_{13}C_9 = 715$ ways.

Selecting a Dozen Donuts Revisited

Example

In how many ways can one dozen donuts be selected if the store offers 9 varieties of donuts, and *at least one of each type* of donut must be selected?

Place one of each type of donut in the box. This leaves us with 3 donuts to select from 9 available types.

This can be done in ${}_{3+9-1}C_3 = {}_{11}C_3 = 165$ ways.

Selecting a Dozen Donuts Revisited

Example

In how many ways can one dozen donuts be selected if the store offers 5 varieties of donuts, and no order can contain *more than* 8 chocolate donuts?

There are a total of ${}_{12+5-1}C_{12} = {}_{16}C_{12} = 1\,820$ possible orders without restrictions.

There are a total of ${}_{3+5-1}C_3 = {}_7C_3 = 35$ orders that contain at least 9 chocolate donuts.

Therefore, there are $1\,820 - 35 = 1\,785$ orders that contain no more than 8 chocolate donuts.

Selecting a Dozen Donuts Revisited

Example

In how many ways can one dozen donuts be selected if the store offers 6 varieties of donuts, and your order includes *between* 3 and 7 maple donuts?

We place 3 maple donuts in the box, then select the remaining 9 donuts from the available types. This can be done in ${}_{6+9-1}C_9 = {}_{14}C_9 = 2\,002$ ways.

Next we find the number of orders with 8 or more maple donuts. There are ${}_{6+4-1}C_4 = {}_9C_4 = 126$ such orders.

Thereore, the dozen donuts can be selected in $2\,002 - 126 = 1\,876$ ways.

Questions?