MDM4U: Mathematics of Data Management

Choosing From Distinct Items

Combinations

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Playing the Lottery

In Lotto 6/49, players must select six different integers between 1 and 49. Six numbered balls are drawn, and the jackpot is awarded to a player who has selected all six numbers.

In order to *guarantee* a win, you would need to purchase a ticket for every possibility. How many tickets is this?

There are ${}_{49}P_6$ ways to arrange six of the numbers. . .

. . . but in this case, order does not matter. Picking 4 8 15 16 23 42 is the same as picking 42 16 8 15 4 23.

We have overcounted by 6! ways.

Therefore, there are $\frac{{}_{49}P_6}{6!} = 13\,983\,816$ ways to select the six numbers.

Counting Subsets

How many 6-element subsets can be made from 49 objects?

This is the same as the previous question. There are $\frac{{}_{49}P_6}{6!} = 13\,983\,816$ 6-element subsets.

In general, the number of r-element subsets, taken from a set containing n elements, is given by $\frac{{}_nP_r}{r!}$.

Counting Subsets

A selection of r objects, taken from a collection of n possible objects, where order *does not* matter is known as a *combination*.

Since we "choose" r of n items, ${}_nC_r$ is typically read "n choose r."

Combinations of r Items, Taken From n Distinct Items

Given n distinct items, the number of combinations of r items, denoted ${}_nC_r$ or $\binom{n}{r}$, is $\frac{n!}{r!(n-r)!}$

Counting Subsets

Explanation: The number of arrangements of r of n items is given by ${}_nP_r$.

Since order does not matter, divide ${}_nP_r$ by $r!$.

Therefore, ${}_nC_r = \frac{{}_nP_r}{r!}$.

Since ${}_nP_r = \frac{n!}{(n-r)!}$, ${}_nC_r = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}$.

Combinations

Example

Verify that ${}_{49}C_6 = 13\,983\,816$.

$$\begin{aligned} {}_{49}C_6 &= \frac{49!}{6!(49-6)!} \\ &= \frac{49!}{6!43!} \\ &= \frac{49 \times 48 \times 47 \times 46 \times 45 \times 44}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ &= 13\,983\,816 \end{aligned}$$

Combinations

Example

Evaluate ${}_6C_4$.

$$\begin{aligned} {}_6C_4 &= \frac{6!}{4!(6-4)!} \\ &= \frac{6!}{4!2!} \\ &= \frac{6 \times 5 \times 4!}{4!2!} \\ &= \frac{6 \times 5}{2} \\ &= 15 \end{aligned}$$

Combinations

Without calculating, what is the value of ${}_{100}C_1$?

There are 100 ways to choose one item. Therefore, ${}_nC_1 = n$.

What about the value of ${}_{50}C_0$?

There is always one way to choose no items – simply do not select any! Therefore, ${}_nC_0 = 1$.

These are useful to remember, as they occur frequently.

Combinations

Example

From a standard deck of cards, how many five-card hands are possible?

There are 52 cards in a deck, from which five are selected.

Therefore, there are ${}_{52}C_5 = \frac{52!}{5!(52-5)!} = 2\,598\,960$ possible five-card hands.

Combinations

Example

From a standard deck of cards, how many five-card hands are made entirely of spades?

There are 13 spades in the deck, from which five are selected.

Therefore, there are ${}_{13}C_5 = \frac{13!}{5!(13-5)!} = 1\,287$ possible hands consisting entirely of spades.

Combinations

Sometimes it is necessary to break down the question into two or more pieces that involve combinations.

For example, we may want to choose some items from one group, then choose other items from a second group.

In this case, we can use the Fundamental Counting Principle or Rule of Sum as appropriate.

Combinations

Example

Your name, along with nine others, is put into a hat. Four names are randomly drawn, without replacement. How many four-name draws include your name?

Your name can be selected in ${}_1C_1 = 1$ way.

The remaining three names can be drawn in ${}_9C_3 = 84$ ways.

Thus, there are ${}_1C_1 \times {}_9C_3 = 1 \times 84 = 84$ four-name draws that include your name.

Combinations

Example

In how many ways can a six-member committee of three men and three women be made from a group of eight men and seven women?

There are ${}_8C_3$ ways to choose the men.

There are ${}_7C_3$ ways to choose the women.

Therefore, the number of ways to create the six-member committee is ${}_8C_3 \times {}_7C_3 = 1\,960$.

Combinations

Example

How many six-member committees of three men and three women, made from a group of eight men and seven women, contain both Bob and Alice?

Since Bob and Alice must be part of the committee, we need only select two additional men from the remaining seven, and two additional women from the remaining six.

There are ${}_7C_2$ ways to choose the men, and ${}_6C_2$ ways to choose the women.

Therefore, there are ${}_7C_2 \times {}_6C_2 = 315$ possible committees.

Combinations

Example

A car dealership sells 9 different sedans, 6 trucks and 4 sports cars. How many ways are there to select a dozen vehicles for its showroom, including *at least one* of each type of vehicle, if order is unimportant?

First, select one of each type of vehicle. This can be done in ${}_9C_1 \times {}_6C_1 \times {}_4C_1$ ways.

Next, select the other 9 vehicles from the remaining 16. This can be done in ${}_{16}C_9$ ways.

Thus, there are ${}_9C_1 \times {}_6C_1 \times {}_4C_1 \times {}_{16}C_9 = 2\,471\,040$ ways to select the cars.

Combinations

As a comparison, how many combinations of a dozen cars are there with no restrictions?

There are ${}_{19}C_{12} = 50\,388$ ways to select 12 cars from 19.

Uh oh.

Our value of 2 471 040 is approximately 49 times too big, so where did we go wrong?

Let's look closer at the expression we used to calculate the number of combinations.

Combinations

$$\underbrace{{}_9C_1}_{\text{1 sedan}} \times \underbrace{{}_6C_1}_{\text{1 truck}} \times \underbrace{{}_4C_1}_{\text{1 sport}} \times \underbrace{{}_{16}C_9}_{\text{remaining}}$$

Choosing the sedans occurs twice – once for the initial selection (${}_9C_1$) and once for the remaining cars (${}_{16}C_9$).

Let's assume, for example, that a blue sedan is chosen initially, then a silver sedan chosen later.

In a similar fashion, we might choose a silver sedan initially, followed later by a blue sedan.

These options are counted twice, even though they might result in the same selection of cars. The same applied when selecting the trucks and the sports cars.

Combinations

To fix this, we can use an indirect method whereby we eliminate all selections containing zero of each type of car.

Criteria	Number of Selections
0 sedans	${}_{10}C_{12}$ =not possible
0 trucks	${}_{13}C_{12} = 13$
0 sports	${}_{15}C_{12} = 455$

Therefore, there are $50\,388 - 13 - 455 = 49\,920$ selections containing at least one of each type of vehicle.

Questions?