MDM4U: Mathematics of Data Management

The Birthday Problem

... and Related Questions

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The Birthday Problem

What is the minimum number of people needed to guarantee a 50% chance that *at least two* people share the same birthday?

Would you believe it takes only 23 people?

How about a 99% chance?

57 people will do.

The Birthday Problem

How is it possible that only 57 people are required to guarantee a 99% chance of sharing the same birthday? Shouldn't it be closer to 365?

Let's try a simpler case with only two people.

We can use an indirect method here, as it is easier to find the probability that they *do not* share a birthday.

Pick any birthday for the first person. This leaves 364 remaining days.

Thus, there is a $\frac{364}{365}$ probability that second was born on a different day.

Therefore, the chance of them sharing the *same* birthday must be $1 - \frac{364}{365} = \frac{1}{365} \approx 0.003$.

The Birthday Problem

Now try the case with three people.

There is a $\frac{364}{365}$ probability that the second person has a different birthday, and a $\frac{363}{365}$ probability that the third has a different birthday from the previous two.

Indirectly, the chance of the three sharing a birthday is

$$1 - \underbrace{\frac{364}{365} \times \frac{363}{365}}_{\text{2 values}} \approx 0.008.$$

The Birthday Problem

In the case of 23 people, we can extend the pattern.

$$1 - \underbrace{\frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{343}{365}}_{\text{22 values}} \approx 0.507$$

This can be extended for use with n people.

The Birthday Problem (Product Rule Approach)

Let $P(n)$ be the probability of at least two of n people having the same birthday, assuming 365 days per year. The probability of this is given by

$$1 - \underbrace{\frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{365-(n-1)}{365}}_{n\text{-1 values}}$$

The Birthday Problem

This method has one major disadvantage, in that it is very tedious. Imagine trying to calculate the probability with a large number of people, such as 50.

The probability of 50 people sharing the same birthday would be $1 - \underbrace{\frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{316}{365}}_{\text{49 values}} \approx 0.970$

That would take a long time to calculate!

Here is a "better" method...

The Birthday Problem

Recall that for 2 people, the probability of them sharing the same birthday is $1 - \frac{364}{365}$.

Another way to write this is $1 - \frac{365}{365} \times \frac{364}{365}$.

In this case, the numerators simplify to ${}_{365}P_2$ and the denominators to 365^2.

Therefore, the probability of 2 people sharing the same birthday is $1 - \frac{{}_{365}P_2}{365^2}$.

The Birthday Problem

In the case of 3 people, the probability of them sharing the same birthday is $1 - \frac{364}{365} \times \frac{363}{365}$.

Another way to write this is $1 - \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365}$.

In this case, the numerators simplify to ${}_{365}P_3$ and the denominators to 365^3.

Therefore, the probability of 3 people sharing the same birthday is $1 - \frac{{}_{365}P_3}{365^3}$.

The Birthday Problem

This gives us an alternative method of calculating the probability of n people sharing the same birthday.

The Birthday Problem (Permutations Approach)

Let $P(n)$ be the probability of at least two of n people having the same birthday, assuming 365 days per year. The probability of this is given by $P(n) = 1 - \frac{{}_{365}P_n}{365^n}$

We can verify our earlier result for 23 people using this formula.

$P(n) = 1 - \frac{{}_{365}P_{23}}{365^{23}} \approx 0.507$.

The Birthday Problem

Your Turn

Determine the probability that at least two attendees at a meeting of twelve people have the same birthday.

$P(12) = 1 - \frac{{}_{365}P_{12}}{365^{12}} \approx 0.167$.

The Birthday Problem

There is one disadvantage to this approach, however...

Verify that 57 people produce a 99% chance of two people sharing a birthday.

Chances are that your calculator cannot handle such large values. On my graphing calculator, I can handle values only up to $n = 39$. In this case, ${}_{365}P_{39} \approx 1.035 \times 10^{99}$.

In this case, more powerful software is required to handle extremely large values.

Related Problems

There are many other types of problems that can be solved using the same techniques used to solve the birthday problem.

Example

An popular chain sells 31 flavours of ice cream. Determine the probability that of the first 10 customers that purchase a single scoop of ice cream, at least two order the same flavour.

$P(10) = 1 - \frac{{}_{31}P_{10}}{31^{10}} \approx 0.804$.

Related Problems

Your Turn

Twenty odd, two-digit integers are selected at random. What is the probability that all of the numbers are not distinct?

There are $9 \times 5 = 45$ odd, two-digit integers.

Therefore, $P(20) = 1 - \frac{{}_{45}P_{20}}{45^{20}} \approx 0.993$.

Questions?