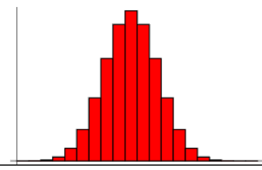
MDM4U: Mathematics of Data Management

Likelihood of Successes and Failures

Binomial Probability Distributions

J. Garvin



Binomial Probability Distributions

Recap

A spinner with three equal sectors, labelled 1-3, is spun three times. What is the probability that a 1 is spun exactly twice?

There are $3^3 = 27$ possible outcomes. Group these according to the number of 1s.

1s	Outcomes	Probability
0	222, 223, 232, 233, 322, 323, 332, 333	$\frac{8}{27}$
1	122, 123, 132, . . . , 231, 321, 331	$\frac{12}{27}$
2	112, 113, 121, 131, 211, 311	$\frac{6}{27}$
3	111	$\frac{1}{27}$

The probability of spinning a 1 exactly twice is $\frac{6}{27} = \frac{2}{9}$.

Terminology

The previous example was an example of *repeated trials*.

An process consists of repeated trials if:

- the experiments are identical
- the experiments are independent

If repeated trials can be represented as *successes* and *failures*, we refer to them as *Bernoulli trials*.

Binomial Probability Distributions

Probability in a Binomial Probability Distribution

The probability of x succsses in a binomial probability distribution with n Bernoulli trials with probability p of success and probability q of failure is
$P(x) = {}_nC_x p^x q^{n-x}$

This should be intuitive, since if there are x successes, there must be $n - x$ failures.

According to FCP, the probability of obtaining a particular sequence of x successes and $n - x$ failures is $p^x q^{n-x}$.

The x successes can occur in ${}_nC_x$ ways, giving the equation above.

Binomial Probability Distributions

Example

Verify that the probability of spinning exactly two 1s is $\frac{2}{9}$.

There are $n = 3$ trials, probability $p = \frac{1}{3}$ of success, probability $q = 1 - \frac{1}{3} = \frac{2}{3}$ of failure, and 2 successes.

$$P(2) = {}_3C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{3-2}$$
$$= 3\left(\frac{1}{9}\right)\left(\frac{2}{3}\right)$$
$$= \frac{2}{9}$$

Binomial Probability Distributions

Your Turn

A standard die is rolled five times. Determine the probability that exactly four 3s are rolled.

There are $n = 5$ trials, probability $p = \frac{1}{6}$ of success, probability $q = 1 - \frac{1}{6} = \frac{5}{6}$ of failure, and 4 successes.

$$P(4) = {}_5C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^1$$
$$P(4) = 5\left(\frac{1}{1296}\right)\left(\frac{5}{6}\right)$$
$$= \frac{25}{7776}$$

Binomial Probability Distributions

Example

A fair coin is tossed five times. What is the probability that *at least* two heads are tossed?

To have at least two heads tossed, disregard the cases where no heads are tossed, or when one head is tossed.

$$\begin{aligned} P(X \geq 2) &= 1 - P(0) - P(1) \\ &= 1 - {}_5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 - {}_5C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^4 \\ &= \frac{13}{16} \end{aligned}$$

Expected Value

What is the expected number of heads tossed in the earlier example?

Heads	Probability	$xP(x)$
0	$\frac{1}{32}$	0
1	$\frac{5}{32}$	$\frac{5}{32}$
2	$\frac{10}{32}$	$\frac{20}{32}$
3	$\frac{10}{32}$	$\frac{30}{32}$
4	$\frac{5}{32}$	$\frac{20}{32}$
5	$\frac{1}{32}$	$\frac{5}{32}$

The expected number of heads is $E(X) = \sum_{i=0}^{n} x_i P(x_i) = 2.5$.

Expected Value

There is a formula to calculate the expected value in a binomial probability distribution.

Expected Value in a Binomial Probability Distribution

For a binomial probability distribution with n trials, each with a probability p of success, the expected value is $E(x) = np$.

The proof uses some properties of sums, and is left as an exercise for the determined student.

Check the previous example: $E(X) = 5 \times \left(\frac{1}{2}\right) = 2.5$.

Expected Value

Example

A game consists of rolling a pair of dice ten times. For each sum of 6, 7 or 8 on the dice, the player wins $1. If it costs $5 to play the game, is it fair?

Solution: There are 16 successful outcomes, shown below.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Expected Value

Therefore, the probability of success is $p = \frac{16}{36} = \frac{4}{9}$.

The expected value of a binomial distribution with $n = 10$ and $p = \frac{4}{9}$ is $np = 10 \times \frac{4}{9} = \frac{40}{9}$ or approximately $4.44.

Since the player pays $5 to play the game, (s)he can expect to lose about $0.56 each game. The game is not fair.

Expected Value

Your Turn

Whenever a hockey player gets a breakaway on the opponent's net, he misses the shot 4 times out of 10. If he averages two breakaways per game, what is the expected number of goals that he will score in a season of 75 games?

The probability of failure is $q = \frac{4}{10} = \frac{2}{5}$.

Therefore, the probability of success is $p = 1 - \frac{2}{5} = \frac{3}{5}$.

There are two breakaways per game, for a total of $n = 75 \times 2 = 150$ trials.

Therefore, he can be expected to score $np = 150 \times \frac{3}{5} = 90$ goals.

Questions?