Birthday Dice

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Cumulative Project (Probability)
MDM4U: Data Management

Overview

In this game, the player rolls three identical dice, each with *N* sides, and tries to roll 2 or more of the same value. The number of sides is determined by spinning a spinner. If any two numbers are identical, the player receives a payout that is a multiple of his/her bet. The game has a birthday theme to it – the game is decorated with presents, the spinner is decorated as a cake, and so on.

Materials

The game uses the following materials:

- Spinner divided into 12 equally-sized sectors (Figure 1 below)
- Three 6-sided dice
- Three 10-sided dice
- Three 12-sided dice
- Three 20-sided dice

Game Play and Rules

In order to play, the player must first place a bet. A player may bet any amount of money, with a minimum bet of $1.00 per round. Once his/her bet is placed, the player spins the spinner to determine which set of dice will need to be rolled. There are four different sets of dice (with 6, 10, 12 and 20 sides), as indicated on the spinner in Figure 1 below.

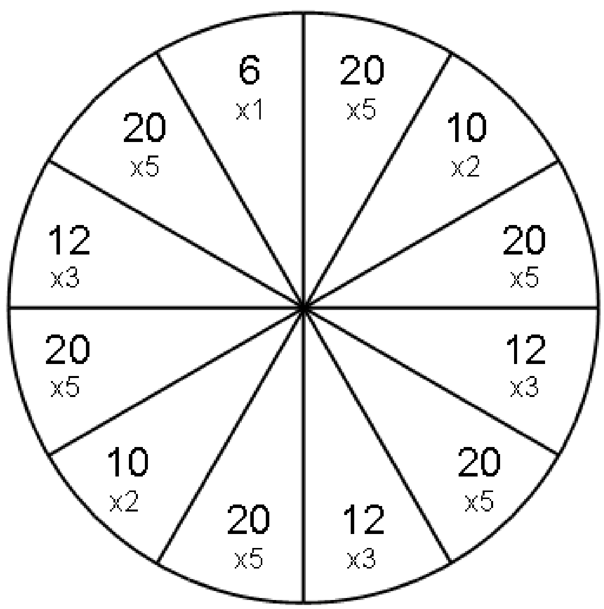

Figure 1: Spinner's 12 Equal Sectors

The player then rolls the three *N*-sided dice. If any two dice have the same value, the player wins a payout equal to the multiplier indicated beneath *N* on the spinner. For example, a player betting $5.00 on the 12-sided dice stands to gain a payout of 3 times the amount, for a *net* payout of $10.00.

The player must place his/her bet before spinning the spinner. After the spin has been made, there cannot be any modifications to the bet.

Theoretical Probabilities and Expected Value

This game is a variation of the classic Birthday Problem. The player must match at least two numbers in order to win. This is equivalent to 2 or more of M people sharing the same birthday within a range of N possible dates. This is given by the formula

$$P(x \geq 2) = 1 - \frac{{}_N P_M}{N^M}.$$

The range of values, N, is determined by spinning the spinner. Spinning the spinner and rolling the dice are independent events, so the Product Rule for independent events is used to calculate the probability of each outcome.

The probabilities of landing on any given number on the spinner are summarized in Table 1.

# Sides (N)	# Sectors	$P(N)$
6	1	$\frac{1}{12}$
10	2	$\frac{2}{12}=\frac{1}{6}$
12	3	$\frac{3}{12}=\frac{1}{4}$
20	6	$\frac{6}{12}=\frac{1}{2}$

Table 1: Probabilities of Spinning N

To analyze the probabilities of winning and losing, the outcomes have been split into winning scenarios (Table 2) and losing scenarios (Table 3). All scenarios are based on a $1.00 buy-in. As the player can bet any amount per round, payouts must be scaled appropriately for higher dollar amounts.

# Sides (N)	$P(N)$	$P(M)$	$P(x)=P(N)\times P(M)$	Payout	Net Gain (x)	$x\,P(x)$ (approx.)
6	$\frac{1}{12}$	$1-\frac{{}_6P_3}{6^3}=\frac{4}{9}$	$\frac{1}{12}\times\frac{4}{9}=\frac{1}{27}$	$1.00	$0.00	$0.0000
10	$\frac{1}{6}$	$1-\frac{{}_{10}P_3}{10^3}=\frac{7}{25}$	$\frac{1}{6}\times\frac{7}{25}=\frac{7}{150}$	$2.00	$1.00	$0.0467
12	$\frac{1}{4}$	$1-\frac{{}_{12}P_3}{12^3}=\frac{17}{72}$	$\frac{1}{4}\times\frac{17}{72}=\frac{17}{288}$	$3.00	$2.00	$0.1181
20	$\frac{1}{2}$	$1-\frac{{}_{20}P_3}{20^3}=\frac{29}{200}$	$\frac{1}{2}\times\frac{29}{200}=\frac{29}{400}$	$5.00	$4.00	$0.2900
						$0.4547

Table 2: Expected Payout per $1 Bet (Winning Scenarios)

# Sides (N)	$P(N)$	$P(M)$	$P(x)=P(N)\times P(M)$	Payout	Net Loss (x)	$x\,P(x)$ (approx.)
6	$\frac{1}{12}$	$\frac{{}_6P_3}{6^3}=\frac{5}{9}$	$\frac{1}{12}\times\frac{5}{9}=\frac{5}{108}$	\$0.00	-\$1.00	-\$0.0463
10	$\frac{1}{6}$	$\frac{{}_{10}P_3}{10^3}=\frac{18}{25}$	$\frac{1}{6}\times\frac{18}{25}=\frac{3}{25}$	\$0.00	-\$1.00	-\$0.1200
12	$\frac{1}{4}$	$\frac{{}_{12}P_3}{12^3}=\frac{55}{72}$	$\frac{1}{4}\times\frac{55}{72}=\frac{55}{288}$	\$0.00	-\$1.00	-\$0.1910
20	$\frac{1}{2}$	$\frac{{}_{20}P_3}{20^3}=\frac{171}{200}$	$\frac{1}{2}\times\frac{171}{200}=\frac{171}{400}$	\$0.00	-\$1.00	-\$0.4275
						-\$0.7848

Table 3: Expected Net Payout per \$1 Bet (Losing Scenarios)

All possibilities are accounted for, as the sum of the probabilities is 1.

$$\frac{1}{27}+\frac{7}{150}+\frac{17}{288}+\frac{29}{400}+\frac{5}{108}+\frac{3}{25}+\frac{55}{288}+\frac{171}{400}=1$$

The expected payout is the sum of the net payouts for the winning and losing scenarios.

$$E(X)\approx 0.4547+(-0.7848)\approx -0.33$$

Therefore, on average a player can expect to lose approximately 33 cents per dollar bet. Higher bets scale this net loss linearly – on average, a \$2.00 bet will lose approximately 66 cents, and so forth.

Methodology

The probabilities have been designed so that the game is difficult to win, but not impossible. In the case of the 6-sided dice, the player has a 4 in 9 chance of winning, which is fairly good compared to typical casino odds. Even in the worst case scenario (the 20-sided dice), 29 of every 200 players should expect to win. This makes the game more appealing than one in which the player always loses.

Payouts were chosen to make certain outcomes more appealing to the player, but still result in a net profit for the House. As the probability of winning using the 20-sided dice is far lower than that of the 6-sided dice, a payout of 5 times the bet may alleviate some concerns about the probabilities involved, and a player may feel the risk is justifiable.

Conclusion

An expected loss of 33 cents per dollar bet means that the game is not fair, and is biased in the House's favour. On average, the player stands to lose this game regardless of the amount wagered or the dice used. In all situations, the probability of winning is less than that of losing, and even the high payouts result in an overall expected value that is biased against the player.