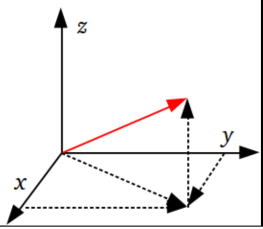
MCV4U: Calculus & Vectors

Applications of the Dot and Cross Products

Part 2: Work and Torque

J. Garvin



ALGEBRAIC VECTORS

Work

Work is when a force is applied to an object, causing it to move.

It is defined as the the product of the object's displacement and the component of the force applied along the line of displacement.

Work can be calculated using the dot product, using either geometric or algebraic vectors.

ALGEBRAIC VECTORS

Work

Work

The work, W, done by a force is given by $W = \vec{F} \cdot \vec{d} = |\vec{F}| \cdot |\vec{d}| \cos\theta$, where $|\vec{F}|$ is the magnitude of the applied force (in Newtons), $|\vec{d}|$ is the magnitude of the object's displacement (in metres), and θ is the angle (in degrees) between the force and the displacement vectors.

The standard unit of work is the Joule (J), or N·m.

ALGEBRAIC VECTORS

Work

Example

Calculate the work done if a sled is pulled forward 50 m along a frictionless surface by a force of 250 N at an angle of 35° to the horizontal.

$$\begin{aligned} W &= (250)(50)\cos 35^\circ \\ &\approx 10\,239 \text{ J} \end{aligned}$$

ALGEBRAIC VECTORS

Work

Example

It takes 12 000 J of work to pull a sled 200 m with a 150 N force. Determine the angle of the rope with the horizontal.

$$\begin{aligned} 12\,000 &= (150)(200)\cos\theta \\ \theta &= \cos^{-1}\left(\frac{12\,000}{(150)(200)}\right) \\ &\approx 66^\circ \end{aligned}$$

ALGEBRAIC VECTORS

Work

Example

A 30 kg box is placed 10 m up a ramp that is inclined at 23° to the horizontal. Calculate the work done by the force of gravity as the box slides down to the bottom of the ramp.

The force of gravity acting downward on the box is $30 \times 9.8 = 294$ N.

The angle between the displacement down the ramp and the force of gravity is $90^\circ - 23^\circ = 67^\circ$.

The work done by gravity, then, is

$$\begin{aligned} W &= (294)(10)\cos 67^\circ \\ &\approx 1149 \text{ J} \end{aligned}$$

Work

Example

A force of 20 N, applied in the direction of $\vec{m} = (1, 2)$, moves an object from $P(3, 2)$ to $Q(6, 15)$. Determine the work done.

The vector representing the applied force, $\vec{f}$, must have the same direction as $\vec{m}$, and a magnitude of 20.

Begin by representing $\vec{f}$ as a scalar multiple of $\vec{m}$.

$$\begin{aligned} 20 &= k\sqrt{1^2 + 2^2} \\ k &= \tfrac{20}{\sqrt{5}} \\ \vec{f} &= \left(\tfrac{20}{\sqrt{5}}, \tfrac{40}{\sqrt{5}}\right) \\ &= (4\sqrt{5}, 8\sqrt{5}) \end{aligned}$$

Work

The displacement vector, $\vec{d}$, can be calculated via subtraction.

$$\begin{aligned} \vec{d} &= (6 - 3, 15 - 2) \\ &= (3, 13) \end{aligned}$$

Now use the dot product for algebraic vectors to calculate the work done.

$$\begin{aligned} W &= (4\sqrt{5}, 8\sqrt{5}) \cdot (3, 13) \\ &= 12\sqrt{5} + 104\sqrt{5} \\ &= 116\sqrt{5} \\ &\approx 259.4 \text{ J} \end{aligned}$$

Work

The last example illustrates an alternate method of calculating work, given algebraic vectors.

Work

The work, W, required to move an object from $P(x_p, y_p, z_p)$ to $Q(x_q, y_q, z_q)$ using some force, $\vec{f}$, in the direction of $\vec{m} = (x_m, y_m, z_m)$ is $\frac{\vec{m}}{|\vec{m}|}|\vec{f}| \cdot (x_q - x_p, y_q - y_p, z_q - z_p)$.

Torque

The turning effect of a force around a fixed point is known as *torque*.

Any time a force is applied to an object that results in some rotation about a point (pedalling a bicycle, opening a door, turning a wrench), a force perpendicular to the plane containing the applied force is produced.

If $\vec{r}$ is the vector representing the radius of the rotation, and f is the vector representing the applied force, then the torque is $\vec{\tau} = \vec{r} \times \vec{f}$.

The magnitude, $|\vec{\tau}| = |\vec{r} \times \vec{f}|$ is a scalar measure of the overall twisting effect.

Like work, torque is measured in Joules.

Torque

Example

A bolt is tightened using a 20 N force, applied at an angle of 60° to the end of a wrench that is 30 cm long. Calculate the magnitude of the torque about its point of rotation.

Since Joules are N·m, convert cm to m first.

$$\begin{aligned} |\vec{\tau}| &= (0.3)(20)\sin 60^\circ \\ &\approx 5.2 \text{ J} \end{aligned}$$

Torque

Your Turn

A bicycle pedal, 20 cm in length, has a 50 N force applied to it at an angle of 45°. Determine the magnitude of the torque.

$$\begin{aligned} |\vec{\tau}| &= (0.2)(50)\sin 45^\circ \\ &= \frac{10}{\sqrt{2}} \\ &= 5\sqrt{2} \\ &\approx 7.1 \text{ J} \end{aligned}$$

Questions?