MCV4U: Calculus & Vectors

Vector/Parametric Equations of a Plane

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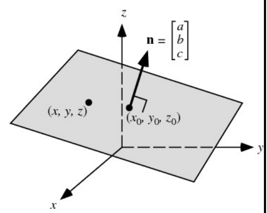
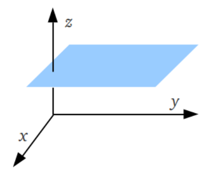


Planes

A *plane* is a flat surface that extends infinitely in all directions.

While lines are typically denoted using the letter L, planes are commonly denoted using the symbol π (not to be confused with the constant).

Planes are often drawn as parallelograms.

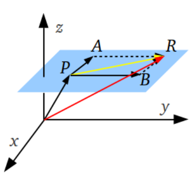


Planes

While it takes two points to uniquely define a line, it takes three non-collinear points to define a plane.

Let these three points be $A(x_a, y_a, z_a)$, $B(x_b, y_b, z_b)$ and $P(x_p, y_p, z_p)$, and construct vectors $\vec{PA}$ and $\vec{PB}$.

The vectors $\vec{PA}$ and $\vec{PB}$, when multiplied by scalars s and t, can be used to reach any point R on the plane using vector addition.



Planes

Let $\vec{OP}$ represent a position vector for point P, and $\vec{OR}$ a position vector for point R.

Then, $\vec{OR} = \vec{OP} + s\vec{PA} + t\vec{PB}$.

Vector Equation of a Plane

The vector equation of a plane is $\vec{r} = (x_p, y_p, z_p) + s(x_a, y_a, z_a) + t(x_b, y_b, z_b)$, where $P(x_p, y_p, z_p)$ is a point on the plane, $\vec{a} = (x_a, y_a, z_a)$ and $\vec{b} = (x_b, y_b, z_b)$ are two non-collinear vectors, and $s, t \in R$.

Planes

Example

Determine the vector equation of the plane containing points $A(1, 0, -2)$, $B(3, 1, 5)$ and $C(-4, 2, 0)$.

Two non-collinear vectors are $\vec{AB} = (2, 1, 7)$ and $\vec{AC} = (-5, 2, 2)$.

A possible vector equation is $\vec{r} = (-4, 2, 0) + s(2, 1, 7) + t(-5, 2, 2)$.

Although different equations are possible, the plane containing these three points is unique, much like a line passing through two points is unique.

Planes

Example

Determine the vector equation of the plane containing the point $P(5, -1, 0)$ and the line $\vec{r} = (3, 1, 1) + k(2, -1, 3)$.

One vector in the plane is the direction vector for the line, $\vec{m} = (2, -1, 3)$.

A second non-collinear vector is $\vec{v} = (3 - 5, 1 - (-1), 1 - 0) = (-2, 2, 1)$.

A possible vector equation is $\vec{r} = (5, -1, 0) + s(2, -1, 3) + t(-2, 2, 1)$.

Note that this will only work when the point is *not* on the line, since all given points would be collinear.

Planes

The parametric equations of a plane can be determined by separating the x-, y- and z-components of the vector equation.

Parametric Equations of a Plane

The parametric equations of a plane are $x = x_p + sx_a + tx_b$, $y = y_p + sy_a + ty_b$ and $z = z_p + sz_a + tz_b$, where $P(x_p, y_p, z_p)$ is a point on the plane, $\vec{a} = (x_a, y_a, z_a)$ and $\vec{b} = (x_b, y_b, z_b)$ are two non-collinear vectors, and $s, t \in R$.

Planes

Example

Determine the parametric equations of the plane that contains points $A(1, 3, -1)$, $B(2, 2, 0)$ and $C(3, 1, -2)$.

Two non-collinear vectors are $\vec{AB} = (1, -1, 1)$ and $\vec{AC} = (2, -2, -1)$.

The parametric equations are $x = 1 + s + 2t$, $y = 3 - s - 2t$ and $z = -1 + s - t$.

Again, there are an infinite number of equations that can define the same, unique plane.

Planes

Example

Determine the parametric equations of the plane that contains the lines $\vec{r_1} = (1, 1, 3) + j(-1, 5, 2)$ and $\vec{r_2} = (4, -2, 0) + k(-1, 5, 2)$.

The lines are parallel, since their direction vectors are the same, but not coincident. Therefore, we are guaranteed to find three non-collinear points.

Two points on the plane are $(1, 1, 3)$ and $(4, -2, 0)$, producing vector $\vec{u} = (3, -3, -3)$ in the plane.

A second vector in the plane is the direction vector of the lines, $\vec{m} = (-1, 5, 2)$.

Thus, the parametric equations of the plane are $x = 1 + 3s - t$, $y = 1 - 3s + 5t$ and $z = 3 - 3s + 2t$.

Planes

Example

Determine the point of intersection between the plane $\vec{r} = (6, -2, -3) + s(1, 3, 0) + t(2, 2, -1)$ and the z-axis.

The parametric equations of the plane are $x = 6 + s + 2t$, $y = -2 + 3s + 2t$ and $z = -3 - t$.

The plane will intersect the z-axis at the point $P(0, 0, z_p)$. Using $x = 0$ and $y = 0$, and rearranging the parametric equations for x and y we obtain the following system:

$$\begin{aligned} s + 2t &= -6 \\ 3s + 2t &= 2 \end{aligned}$$

Planes

Solving the system, $s = 4$ and $t = -5$.

Use these parameters to calculate the value of z in its parametric equation.

$$z = -3 - (-5) = 2$$

Therefore, the plane intersects the z-axis at $P(0, 0, 2)$.

Questions?