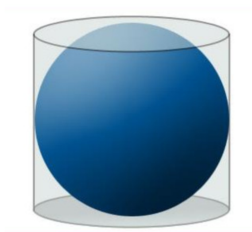
MCV4U: Calculus & Vectors

Related Rates
Part 2

J. Garvin



Related Rates

Example

Two sides of a triangle are 15 and 20 cm long, respectively. How fast is the third side increasing when the angle between the given sides is 60°, and increasing at the rate of 2° per second?

If the third side is a, we want to find $\frac{da}{dt}$.

We are given the rate of change of the angle, $\frac{dA}{dt} = 2$.

The sides are related by the Cosine Law,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cdot \cos A \\ &= 15^2 + 20^2 - 2 \cdot 15 \cdot 20 \cos A \\ &= 625 - 600 \cos A \end{aligned}$$

Related Rates

Implicitly differentiating the Cosine Law,

$$\begin{aligned} 2a \cdot \frac{da}{dt} &= 600 \sin A \cdot \frac{dA}{dt} \\ \frac{da}{dt} &= \frac{300 \sin A \cdot 2}{a} \\ &= \frac{600 \sin A}{a} \end{aligned}$$

When $A = 60°$, $\sin A = \frac{\sqrt{3}}{2}$ and $\cos A = \frac{1}{2}$.

Thus, $a^2 = 625 - 600 \cdot \frac{1}{2} = 325$, so $a = \sqrt{325} = 5\sqrt{13}$.

We can evaluate the derivative,

$$\begin{aligned} \frac{da}{dt} &= \frac{600 \cdot \frac{\sqrt{3}}{2}}{5\sqrt{13}} \\ &= \frac{60\sqrt{39}}{13} \text{ cm/s} \end{aligned}$$

Related Rates

Example

A pile of sand in the shape of a cone has a radius twice its height. The pile grows at a rate of 5 m^3/s. How fast is the height increasing when the radius is 20 m?

We want to know $\frac{dh}{dt}$, and are given $\frac{dV}{dt} = 5$.

Since the radius is twice the height, $r = 2h$.

The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$, or $V = \frac{4}{3}\pi h^3$ after substitution.

Therefore, $\frac{dV}{dh} = 4\pi h^2$.

Related Rates

Using the chain rule,

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dh} \cdot \frac{dh}{dt} \\ 5 &= 4\pi h^2 \cdot \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{5}{4\pi h^2} \end{aligned}$$

When the radius is 20 m, the height is 10 m. Substitute into the above equation.

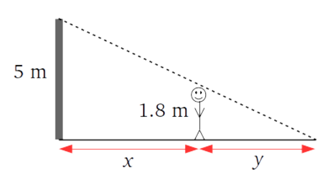
$$\begin{aligned} \left.\frac{dh}{dt}\right|_{h=10} &= \frac{5}{4\pi \cdot 10^2} \\ &= \frac{1}{80\pi} \text{ m/s} \end{aligned}$$

Related Rates

Example

A 1.8 m tall man casts a shadow as he walks away from a 5 m tall lamp post at a rate of 2 m/s. How fast is the shadow growing?

A diagram of the scenario is below, where x is the distance from the man to the lamp post and y the distance from the man to the tip of his shadow.



Related Rates

Using similar triangles,

$$\frac{1.8}{y} = \frac{5}{x+y}$$
$$1.8x + 1.8y = 5y$$
$$1.8x = 3.2y$$

Using implicit differentiation,

$$1.8\frac{dx}{dt} = 3.2\frac{dy}{dt}$$
$$\frac{dy}{dt} = \frac{9}{16} \cdot \frac{dx}{dt}$$

Since we are given $\frac{dx}{dt} = 2$,

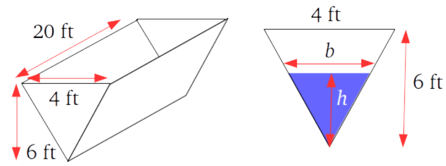
$$\frac{dy}{dt} = \frac{9}{16} \cdot 2$$
$$= \frac{9}{8} \text{ m/s}$$

Related Rates

Example

A triangular trough is 4 ft wide at the top, 6 ft deep, and 20 ft long. If it is filled with water at a rate of 10 ft^3/min, how fast is the water level rising when the water is 2 ft deep?

A diagram of the scenario is below, with a cross-section shown at right. We want $\frac{dh}{dt}$.



Related Rates

Using similar triangles,

$$\frac{b}{4} = \frac{h}{6}$$
$$b = \frac{2}{3}h$$

Substitute into the formula for the volume of the trough,

$$V = \frac{1}{2}bh\ell$$
$$= \frac{1}{2} \cdot \frac{2}{3}h \cdot h \cdot 20$$
$$= \frac{20}{3}h^2$$

Differentiate,

$$\frac{dV}{dt} = \frac{20}{3} \cdot 2h \cdot \frac{dh}{dt}$$

Related Rates

We are given $\frac{dV}{dt} = 10$ and $h = 2$, so substitute these values into the equation.

$$10 = \frac{20}{3} \cdot 2 \cdot 2 \cdot \frac{dh}{dt}$$
$$= \frac{80}{3}\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{3}{8} \text{ ft/min}$$

Questions?