MCV4U: Calculus & Vectors

Rates of Change

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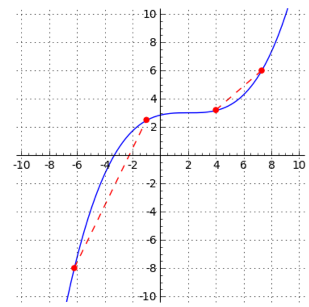
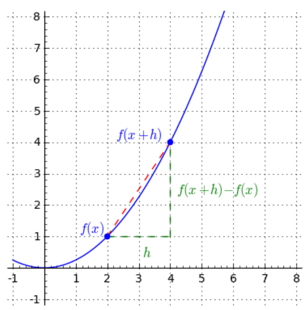
Slope of a Secant

Recall that a *secant* is a line segment that connects two points on a curve, such as the two secants below.



Slope of a Secant

The slope of the secant depends on the magnitude of the interval, h, over which it is taken.



Difference Quotient

Recall that slope is defined as "rise over run".

$$\begin{aligned} m_{\text{secant}} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{f(x+h)-f(x)}{(x+h)-x} \\ &= \frac{f(x+h)-f(x)}{h} \end{aligned}$$

This formula is known as the *difference quotient*.

Difference Quotient

The difference quotient, $\dfrac{f(x+h)-f(x)}{h}$, gives the slope of the secant over the interval $[x, x+h]$.

Difference Quotient

Example

Determine the slope of the secant to $f(x) = x^3 - 5$ on the interval $[-1, 4]$.

The magnitude of the interval is $h = 4 - (-1) = 5$.

When $x = -1$, $f(-1) = (-1)^3 - 5 = -6$.

When $x = 4$, $f(4) = (4)^3 - 5 = 59$.

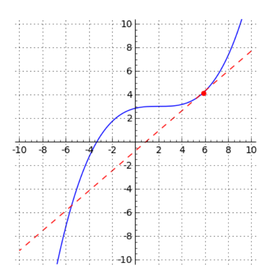
Substitute $h = 4 - (-1) = 5$, $f(-1) = -6$ and $f(4) = 59$ into the difference quotient.

$$\begin{aligned} m_{\text{secant}} &= \frac{59-(-6)}{5} \\ &= 13 \end{aligned}$$

The slope of the secant is 13 on the interval $[-1, 4]$.

Slope of a Tangent

A *tangent* is a line that "just touches" a curve, such as the tangent at $x = 6$ below.



Note that a tangent *may* cross a function at other points.

Slope of a Tangent

As the interval, h, of the secant gets smaller, the slope of the secant better approximates the behaviour of the function at a given point P.

If h is sufficiently small, the slope of the secant will approach the slope of the tangent at P.

In previous courses, you have estimated the slope of the tangent by using a very small interval, such as $h = 0.00001$.

While this is "good enough" for many functions, a more precise method is better.

Difference Quotient (Again)

To calculate the slope of the tangent at a given point, we need to evaluate the difference quotient as $h \to 0$.

Since setting $h = 0$ will result in division by zero, it is necessary to rewrite the difference quotient so that this restriction is eliminated.

Fortunately, if all steps are done correctly, this should happen naturally.

Difference Quotient (Again)

Example

Determine the slope of the tangent to $f(x) = x^2 - 4$ at $x = 3$.

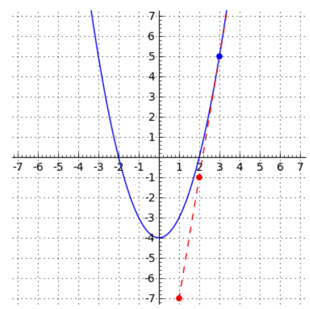
When $x = 3$, $f(3) = 3^2 - 4 = 5$, so the point of tangency is $(3, 5)$.

Substitute $x = 3$ and $f(3) = 5$ into the difference quotient.

$$\begin{aligned} m_{\text{tangent}} &= \frac{(3+h)^2 - 4 - 5}{h} \\ &= \frac{9 + 6h + h^2 - 9}{h} \\ &= \frac{h(6+h)}{h} \\ &= 6 + h \end{aligned}$$

Difference Quotient (Again)

As $h \to 0$, $6 + h \to 6$. Therefore, the slope of the tangent to $f(x) = x^2 - 4$ at $(3, 5)$ is 6.



Rates of Change

Recall that a function's *rate of change* describes how changes by the independent variable cause changes of the dependent variable.

Graphically, a function's rate of change is given by the slope of a secant (average rate of change) or of a tangent (instantaneous rate of change).

By using the difference quotient, we can determine a function's rate of change in either case.

Rates of Change

Example

A ball is kicked into the air, and its height, in metres, after t seconds is given by $h(t) = -4.9t^2 + 29.4t$. Determine:

- its average velocity in the first second,
- its average velocity between 1 and 3 seconds, and
- the instantaneous velocity at 4 seconds.

The first second occurs between $t = 0$ and $t = 1$.

When $t = 0$, $h(0) = 0$. When $t = 1$, $h(1) = 24.5$. The interval is $h = 1$.

The average velocity is $\frac{24.5 - 0}{1} = 24.5$ m/s.

Rates of Change

For the interval between the first and third second, $h(1) = 24.5$, $h(3) = 44.1$, and $h = 2$.

The average velocity is $\dfrac{44.1 - 24.5}{2} = 9.8$ m/s.

Note that the velocity is different that in the first case because of three reasons:

- the intervals have different starting/ending points,
- the magnitudes of the interval are different, and
- the behaviour of the function is different on the given intervals.

Rates of Change

Since the instantaneous velocity is given by the slope of the tangent, we substitute $t = 4$ and let $h \to 0$.

$$\begin{aligned}\text{vel} &= \frac{[-4.9(4+h)^2 + 29.4(4+h)] - [-4.9(4)^2 + 29.4(4)]}{h} \\ &= \frac{[-4.9(16+8h+h^2) + 117.6 + 29.4h] - [-78.4 + 117.6]}{h} \\ &= \frac{-78.4 - 39.2h - 4.9h^2 + 29.4h + 78.4}{h} \\ &= \frac{h(-9.8 - 4.9h)}{h} \\ &= (-9.8 - 4.9h)\end{aligned}$$

As $h \to 0$, $-9.8 - 4.9h \to -9.8$. Therefore, the velocity at 4 seconds is -9.8 m/s, or 9.8 m/s downward.

Questions?