MCV4U: Calculus & Vectors

Algebraic Properties of Limits

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Limits

Recap

What is the value of $\lim_{x \to 2} \frac{x^2 - 9}{x - 2}$ if it exists?

The function is undefined when $x = 2$, but that does not mean that the limit does not exist. We can test values from either side of 2 to estimate the limit.

x	...	1.9	1.99	1.999
$f(x)$	...	53.9	503.99	5003.999

x	...	2.1	2.01	2.001
$f(x)$	...	-45.9	-495.99	-4995.999

Limits

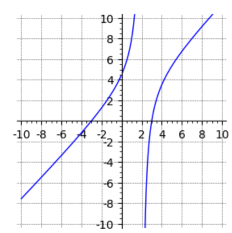
As x approaches 2 from the left, $f(x)$ increase rapidly toward ∞. That is, $\lim_{x \to 2^-} \frac{x^2 - 9}{x - 2}$ appears to be ∞.

As x approaches 2 from the right, $f(x)$ decreases rapidly toward $-\infty$. That is, $\lim_{x \to 2^+} \frac{x^2 - 9}{x - 2}$ appears to be $-\infty$.

Since these values are different, $\lim_{x \to 2} \frac{x^2 - 9}{x - 2}$ does not exist.

Limits

A graph of the function confirms this.



Properties of Limits

While drawing a graph or creating a table of values is useful for simple functions, more complex graphs can be time-consuming or inaccurate.

Instead, we would like to utilize algebraic properties of limits, so that we can evaluate limits based solely on equations of these functions.

Proofs of these are not covered here, as we need to use a more formalized definition of a limit in order to show that the properties hold for all functions.

Properties of Limits

Limit Properties

1. $\lim_{x \to a} c = c$ if c is a constant.
2. $\lim_{x \to a} x = a$.
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$ for some constant c.
4. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$.
5. $\lim_{x \to a} [f(x)g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)]$.
6. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ where $\lim_{x \to a} g(x) \neq 0$.
7. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$ where n is rational.

Properties of Limits

Example

Use limit properties to evaluate $\lim_{x\to 2}(3x-5)$.

$$\begin{aligned}\lim_{x\to 2}(3x-5) &= \lim_{x\to 2}3x - \lim_{x\to 2}5 && (4)\\ &= 3\lim_{x\to 2}x - \lim_{x\to 2}5 && (3)\\ &= 3(2)-5 && (1 \text{ and } 2)\\ &= 1\end{aligned}$$

Properties of Limits

Example

Use limit properties to evaluate $\lim_{x\to 5}\sqrt{3x+1}$.

$$\begin{aligned}\lim_{x\to 5}\sqrt{3x+1} &= \lim_{x\to 5}(3x+1)^{\frac{1}{2}}\\ &= \left[\lim_{x\to 5}(3x+1)\right]^{\frac{1}{2}} && (7)\\ &= \left[3\lim_{x\to 5}x + \lim_{x\to 5}1\right]^{\frac{1}{2}} && (3 \text{ and } 4)\\ &= [3(5)+1]^{\frac{1}{2}} && (1 \text{ and } 2)\\ &= 4\end{aligned}$$

Properties of Limits

Note that in both of the previous examples, the same answer could have come about by simply evaluating each function at the given value.

For example, $\lim_{x\to 5}\sqrt{3x+1} = \sqrt{3(5)+1} = \sqrt{16} = 4$.

Remember that a function does not need to be defined at $x = a$ for us to evaluate $\lim_{x\to a} f(x)$, only that we approach a certain value as we approach a.

In the next session, we will look at some ways in which we can evaluate such functions.

Questions?