MCV4U: Calculus & Vectors

Optimization

Part 1

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Optimization

Optimization is the process of determining the maximum or minimum value of a function, based on some input parameters (usually restrictions on the domain).

In previous courses, you have dealt with optimization for specific, simple cases, such as:

- determining the optimal dimensions of a rectangular prism or cylinder to maximize its volume or minimize its surface area, through empirical measurement,
- determining the maximum area of a rectangle represented by a quadratic relation, by determining the location of the vertex, and
- determining the highest and lowest points on a sinusoidal function, using properties of the function.

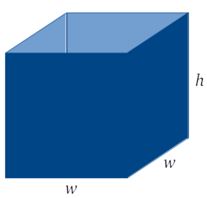
We will now define a more general process for all functions.

Optimization

Example

Determine the dimensions of a square-based, open-topped box that hold a volume of 800 cm^3 and minimizes the amount of material used. What is the surface area?

The box is made of four congruent sides and a square base, where w is the width of the base, and h is the height.



Optimization

A formula for the surface area, S, that we want to minimize is

$$S = w^2 + 4wh$$

Using the formula for volume, we can express the height in terms of the width.

$$\begin{aligned} V &= w^2 h \\ 800 &= w^2 h \\ h &= \frac{800}{w^2} \end{aligned}$$

Optimization

Substitute this into the formula for surface area.

$$\begin{aligned} S &= w^2 + 4w\left(\frac{800}{w^2}\right) \\ &= \frac{w^3 + 3200}{w} \end{aligned}$$

Find the derivative to identify any potential minimums.

$$\begin{aligned} \frac{dS}{dw} &= \frac{3w^2(w) - (w^3 + 3200)}{w^2} \\ &= \frac{2w^3 - 3200}{w^2} \end{aligned}$$

A potential minimum may occur when $2w^3 - 3200 = 0$, or when $w = \sqrt[3]{1600}$ (approx. 11.70).

Optimization

Test the slopes on either side of $x = \sqrt[3]{1600}$ to see if the point is a minimum.

x	11	12
$f'(x)$	$-\frac{538}{121}$	$\frac{7}{4}$
sign	$-$	$+$

Since the slopes change from decreasing to increasing, there is a minimum when $w = \sqrt[3]{1600}$.

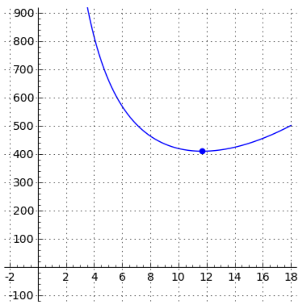
When $w = \sqrt[3]{1600}$, $h = \frac{800}{\left(\sqrt[3]{1600}\right)^2} = 2\sqrt[3]{25}$ (approx. 5.85).

Thus, the dimensions that minimize the material are $\sqrt[3]{1600} \times \sqrt[3]{1600} \times 2\sqrt[3]{25}$ cm.

The surface area is $S = \dfrac{\sqrt[3]{1600}^3 + 3200}{\sqrt[3]{1600}} \approx 410.4$ cm^2.

Optimization

A graph of the surface area shows a minimum at $w = \sqrt[3]{1600}$.

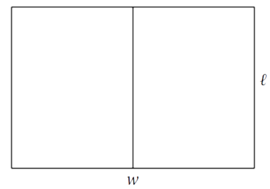


Optimization

Example

60 m of fencing is available to construct a rectangular garden, divided into two sections by a length of fencing perpendicular to its width. If the width can be no greater than 12 m, what dimensions will maximize the garden's area? What is the area?

A diagram is shown, where w is the width and ℓ the length.



Optimization

Since there are 60 m of fencing available, we can develop a formula for the length of the fence, based on its width.

$$60 = 2w + 3\ell$$
$$\ell = 20 - \tfrac{2}{3}w$$

Substitute this into the formula for area, which is the quantity we wish to maximize.

$$\begin{aligned} A &= \ell w \\ &= \left(20 - \tfrac{2}{3}w\right) w \\ &= -\tfrac{2}{3}w^2 + 20w \end{aligned}$$

Optimization

Since there is a restriction on the domain, we need to evaluate the function at the end points.

When $w = 0$, the area will also be 0, which is not practical.

When $w = 12$, the length is $\ell = 20 - \frac{2}{3}(12) = 12$ m and the area is $A = -\frac{2}{3}(12)^2 + 20(12) = 144$ m^2.

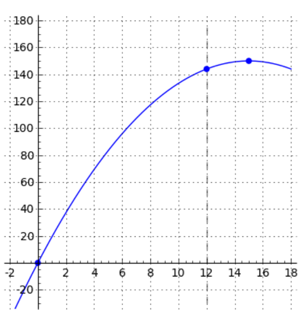
A greater area may occur at a local maximum, provided $0 \le w \le 12$. Find the derivative and identify critical points.

$$\begin{aligned} \tfrac{dA}{dw} &= -\tfrac{4}{3}w + 20 \\ 0 &= -\tfrac{4}{3}w + 20 \\ w &= 15 \end{aligned}$$

Since the critical point falls outside of the specified domain, it is inadmissible. It it *was* admissible, the area would be $A = -\frac{2}{3}(15)^2 + 20(15) = 150$ m^2.

Optimization

A graph of the area shows the maximum when $w = 12$ m.



Questions?