MCV4U: Calculus & Vectors

Operations with Algebraic Vectors

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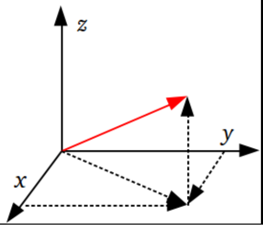

Cartesian Vectors

Recall that any vector $\vec{v}$ can be expressed in terms of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

$$\begin{aligned}\vec{v} &= (x, y, z)\\ &= x\hat{i} + y\hat{j} + z\hat{k}\\ &= (x, 0, 0) + (0, y, 0) + (0, 0, z)\end{aligned}$$

This corresponds to writing a vector as the sum of its components in three-space.

Cartesian Vectors

Example

Express $\vec{u} = (6, 0, -3)$ in terms of its three-dimensional components.

$$\begin{aligned}\vec{u} &= 6\hat{i} + 0\hat{j} - 3\hat{k}\\ &= (6, 0, 0) + (0, 0, 0) + (0, 0, -3)\end{aligned}$$

Example

Express $(5, 0, 0) + (0, -3, 0) + (0, 0, 8)$ as a single vector.

$$\begin{aligned}(5, 0, 0) + (0, -3, 0) + (0, 0, 8) &= 5\hat{i} - 3\hat{j} + 8\hat{k}\\ &= (5, -3, 8)\end{aligned}$$

Cartesian Vectors

As with geometric vectors, we can define operations for use with algebraic vectors.

Specifically, we wish to define operations for vector addition, vector subtraction and scalar multiplication as we did earlier.

To do this, we break down vectors into their three-dimensional components.

Cartesian Vectors

Adding Cartesian Vectors

For vectors $\vec{u} = (x_u, y_u, z_u)$ and $\vec{v} = (x_v, y_v, z_v)$, $\vec{u} + \vec{v} = (x_u + x_v, y_u + y_v, z_u + z_v)$.

$$\begin{aligned}\vec{u} + \vec{v} =& (x_u, y_u, z_u) + (x_v, y_v, z_v)\\ =& (x_u, 0, 0) + (0, y_u, 0) + (0, 0, z_u) +\\ & (x_v, 0, 0) + (0, y_v, 0) + (0, 0, z_v)\\ =& x_u\hat{i} + y_u\hat{j} + z_u\hat{k} + x_v\hat{i} + y_v\hat{j} + z_v\hat{k}\\ =& (x_u + x_v)\hat{i} + (y_u + y_v)\hat{j} + (z_u + z_v)\hat{k}\\ =& (x_u + x_v, y_u + y_v, z_u + z_v)\end{aligned}$$

Cartesian Vectors

The previous definition can be used to establish a rule for subtracting Cartesian vectors.

Subtracting Cartesian Vectors

For vectors $\vec{u} = (x_u, y_u, z_u)$ and $\vec{v} = (x_v, y_v, z_v)$, $\vec{u} - \vec{v} = (x_u - x_v, y_u - y_v, z_u - z_v)$.

$$\begin{aligned}\vec{u} - \vec{v} &= \vec{u} + (-\vec{v})\\ &= (x_u, y_u, z_u) + (-x_v, -y_v, -z_v)\\ &= (x_u + (-x_v), y_u + (-y_v), z_u + (-z_v))\\ &= (x_u - x_v, y_u - y_v, z_u - z_v)\end{aligned}$$

Cartesian Vectors

Example

If $\vec{u} = (3,5,0)$ and $\vec{v} = (4,-8,6)$, determine $\vec{u}+\vec{v}$. and $\vec{u}-\vec{v}$.

$$\begin{aligned}\vec{u}+\vec{v} &= (3,5,0)+(4,-8,6)\\ &= (3+4,5-8,0+6)\\ &= (7,-3,6)\end{aligned}$$

$$\begin{aligned}\vec{u}-\vec{v} &= (3,5,0)-(4,-8,6)\\ &= (3-4,5+8,0-6)\\ &= (-1,13,-6)\end{aligned}$$

Cartesian Vectors

We can also define multiplication by a scalar in terms of Cartesian vectors.

Multiplication By a Scalar

For vector $\vec{v} = (x_v, y_v, z_v)$ and scalar k, $k(x_v, y_v, z_v) = (kx_v, ky_v, kz_v)$.

$$\begin{aligned}k\vec{v} &= k(x_v, y_v, z_v)\\ &= k((x_v,0,0)+(0,y_v,0)+(0,0,z_v))\\ &= k(x_v\hat{i}+y_v\hat{j}+z_v\hat{k})\\ &= kx_v\hat{i}+ky_v\hat{j}+kz_v\hat{k}\\ &= (kx_v,0,0)+(0,ky_v,0)+(0,0,kz_v)\\ &= (kx_v,ky_v,kz_v)\end{aligned}$$

Cartesian Vectors

Example

Given the vectors $\vec{u} = (6,-2,0)$ and $\vec{v} = (-3,8,6)$, find an expression for $3\vec{u}+4\vec{v}$.

$$\begin{aligned}3\vec{u}+4\vec{v} &= 3(6,-2,0)+4(-3,8,6)\\ &= (18,-6,0)+(-12,32,24)\\ &= (6,26,24)\end{aligned}$$

Cartesian Vectors

Example

Using vectors, show that the points $A(5,-1)$, $B(-3,4)$ and $C(13,-6)$ are collinear.

$$\begin{aligned}\vec{AB} &= (-8,5)\\ \vec{BC} &= (16,-10)\\ \vec{BC} &= -2\vec{AB}\end{aligned}$$

Since $\vec{AB}$ and $\vec{BC}$ have opposite directions, the three points must be collinear.

Cartesian Vectors

Example

Parallelogram $OABC$ has one vertex at the origin $O(0,0,0)$ and two opposite vertices at $A(1,-5,2)$ and $C(-3,4,4)$. Determine the coordinates of B.

$$\begin{aligned}\vec{OB} &= \vec{OC}+\vec{CB}\\ \vec{CB} &= \vec{OA}\\ &= (1,-5,2)\\ \vec{OB} &= (-3,4,4)+(1,-5,2)\\ &= (-2,-1,6)\end{aligned}$$

B has coordinates $(-2,-1,6)$.

Questions?