MCV4U: Calculus & Vectors

# Algebraic Vectors In Two- and Three-Space

## Part 2: Magnitudes and Direction Angles

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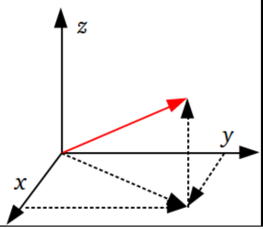







## Magnitudes of Cartesian Vectors In Two-Space

Let $\vec{v}$ be a position vector in the $xy$-plane.

Thus $\vec{v}$ has its tail at the origin and its head at some point $P(x, y)$.

Its magnitude can be calculated using the Pythagorean Theorem, and its direction can be calculated using the tangent ratio.

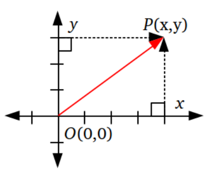




## Magnitudes of Cartesian Vectors In Two-Space

**Magnitude and Direction of a Position Vector In Two-Space**

For any point $P(x, y)$, $|\vec{OP}| = \sqrt{x^2 + y^2}$ and the angle it makes with the positive $x$-axis is given by $\tan^{-1}\left(\frac{y}{x}\right)$.








## Magnitudes of Cartesian Vectors In Two-Space

**Example**

Let $\vec{v} = (5, 12)$. Determine the magnitude of $\vec{v}$ and its direction.

$$\begin{aligned} |\vec{v}| &= \sqrt{5^2 + 12^2} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

$\vec{v}$ has a direction angle of $\tan^{-1}\left(\frac{12}{5}\right) \approx 67°$.





## Magnitudes of Cartesian Vectors In Two-Space

The magnitude of any Cartesian vector can be found using the formula for the distance between two points $P$ and $Q$.

**Magnitude of a Cartesian Vector In Two-Space**

For any two points $P(x_p, y_p)$ and $Q(x_q, y_q)$, $|\vec{PQ}| = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2}$.

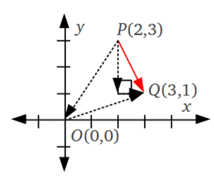
Start by expressing $PQ$ in terms of $OP$ and $OQ$.

$$\begin{aligned} \vec{PQ} &= \vec{PO} + \vec{OQ} \\ &= \vec{OQ} - \vec{OP} \end{aligned}$$





## Magnitudes of Cartesian Vectors In Two-Space




$\vec{OP}$ has a horizontal component of $x_p$ and a vertical component of $y_p$, while $\vec{OQ}$ has a horizontal component of $x_q$ and a vertical component of $y_q$.

The horizontal components have a magnitude of $x_q - x_p$, and the vertical components a magnitude of $y_q - y_p$.

Using the Pythagorean Theorem to combine the components yields $|\vec{PQ}| = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2}$ as required.

## Magnitudes of Cartesian Vectors In Two-Space

**Example**

Given the vectors $\vec{u} = (6, -2)$ and $\vec{v} = (-3, 8)$, determine $|\vec{u} - \vec{v}|$.

$$\begin{aligned} |\vec{u} - \vec{v}| &= \sqrt{(8 - (-2))^2 + (-3 - 6)^2} \\ &= \sqrt{(10)^2 + (-9)^2} \\ &= \sqrt{181} \end{aligned}$$

Note that is equivalent to asking for the magnitude of the line segment between points $U(6, -2)$ and $V(-3, 8)$.





## Magnitudes of Cartesian Vectors In Three-Space

The formulae for magnitudes of vectors in three-space can be extended from those in two-space.

**Magnitude of a Position Vector in Three-Space**

For any point $P(x, y, z)$, $|\vec{OP}| = \sqrt{x^2 + y^2 + z^2}$.

**Magnitude of a Cartesian Vector in Three-Space**

For any two points $P(x_p, y_p, z_p)$ and $Q(x_q, y_q, z_q)$, $|\vec{PQ}| = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2 + (z_q - z_p)^2}$.





## Magnitudes of Cartesian Vectors In Three-Space

**Example**

Determine the magnitude of $\vec{v} = (9, 2, -1)$.

$$\begin{aligned} |\vec{v}| &= \sqrt{9^2 + 2^2 + (-1)^2} \\ &= \sqrt{81 + 4 + 1} \\ &= \sqrt{86} \end{aligned}$$





## Magnitudes of Cartesian Vectors In Three-Space

**Example**

Determine the magnitude of $\vec{v} = \left(\frac{5}{9}, \frac{4}{9}, \frac{2}{9}\right)$.

$$\begin{aligned} |\vec{v}| &= \sqrt{\left(\tfrac{5}{9}\right)^2 + \left(\tfrac{4}{9}\right)^2 + \left(\tfrac{2}{9}\right)^2} \\ &= \sqrt{\tfrac{25}{81} + \tfrac{16}{81} + \tfrac{4}{81}} \\ &= \sqrt{\tfrac{45}{81}} \\ &= \tfrac{\sqrt{5}}{3} \end{aligned}$$





## Magnitudes of Cartesian Vectors In Three-Space

**Alternate Solution**

Determine the magnitude of $\vec{v} = \left(\frac{5}{9}, \frac{4}{9}, \frac{2}{9}\right)$.

Note that $\vec{v} = \frac{1}{9}(5, 4, 2)$.

$$\begin{aligned} |\vec{v}| &= \tfrac{1}{9}\sqrt{(5^2 + 4^2 + 2^2)} \\ &= \tfrac{1}{9}\sqrt{45} \\ &= \tfrac{\sqrt{45}}{9} \\ &= \tfrac{\sqrt{5}}{3} \end{aligned}$$





## Magnitudes of Cartesian Vectors In Three-Space

**Example**

Given the vectors $\vec{u} = (8, -1, 5)$ and $\vec{v} = (4, 2, -6)$, determine $|\vec{u} - \vec{v}|$.

$$\begin{aligned} |\vec{u} - \vec{v}| &= \sqrt{(4 - 8)^2 + (2 - (-1))^2 + (-6 - 5)^2} \\ &= \sqrt{(-4)^2 + 3^2 + (-11)^2} \\ &= \sqrt{146} \end{aligned}$$

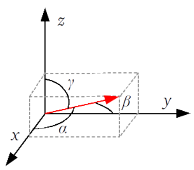
## Direction Angles and Direction Cosines

What about direction vectors in three-space?

Rather than introduce a new system to replace those in place for two-space (NEWS or bearings), we can use the angles relative to the three axes.

**Direction Angles of a Vector**

The direction angles, $\alpha$, $\beta$ and $\gamma$, are the acute or obtuse angles formed between a vector $\vec{v}$ and the $x$-, $y$- and $z$-axes.








## Direction Angles and Direction Cosines

To calculate direction angles, we use the *direction cosines* of the vector.

**Direction Angles of a Vector**

The direction cosines of a vector $\vec{v}$ are:
$\cos\alpha = \frac{x}{|\vec{v}|}$ $\cos\beta = \frac{y}{|\vec{v}|}$ $\cos\gamma = \frac{z}{|\vec{v}|}$

The direction cosines have the property that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.





## Direction Angles and Direction Cosines

**Example**

Determine the direction angles for $\vec{v} = (5, -2, 3)$.

First, determine $|\vec{v}|$.

$$\begin{aligned} |\vec{v}| &= \sqrt{5^2 + (-2)^2 + 3^2} \\ &= \sqrt{38} \end{aligned}$$

Next, use the direction cosines to calculate the angles.

$$\cos\alpha = \frac{5}{\sqrt{38}} \qquad \cos\beta = \frac{-2}{\sqrt{38}} \qquad \cos\gamma = \frac{3}{\sqrt{38}}$$

$$\alpha \approx 36^\circ \qquad \beta \approx 109^\circ \qquad \gamma \approx 61^\circ$$





## Questions?