MCV4U: Calculus & Vectors

Equations of Lines In Three-Space

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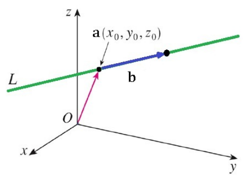

Equations of Lines In Three-Space

Like in two-space, there are several ways to represent a line in three-space.

These include vector, parametric, and symmetric forms.

Unlike in two-space, there are no slope-intercept or scalar forms in three-space.

As we shall see later, a scalar equation in three-space represents a plane.

Vector Equation of a Line

By starting at point $P(x_p, y_p, z_p)$ and moving in the same or opposite direction of a direction vector $\vec{m} = (x_m, y_m, z_m)$, we can reach any point on the line.

This new point will have a position vector equal to the resultant of $\vec{p} + t\vec{m}$, where t is some real value.

This is the same situation developed earlier for the vector form of a line in two-space, extended to three-space.

Vector Equation of a Line In Three-Space

The vector equation of a line in three-space is $\vec{r} = (x_p, y_p, z_p) + t(x_m, y_m, z_m)$, where $\vec{p} = (x_p, y_p, z_p)$ is a position vector of a point on the line, $\vec{m} = (x_m, y_m, z_m)$ is a direction vector for the line, and $t \in R$.

Vector Equation of a Line

Example

Determine the vector equation of the line that passes through $A(2, -1, 5)$ and $B(6, 5, -3)$.

Using $\vec{AB}$ as a direction vector for the line, $\vec{m} = (6 - 2, 5 - (-1), -3 - 5) = (4, 6, -8)$, but any scalar multiple will do, such as $\vec{m} = (2, 3, -4)$.

Using point A, a possible equation for the line is $\vec{r} = (2, -1, 5) + t(2, 3, -4)$.

Using point B, another possible equation is $\vec{r} = (6, 5, -3) + t(2, 3, -4)$.

Vector Equation of a Line

Example

Determine the vector equation of the line, perpendicular to the xy-plane, that passes through $P(1, 2, 3)$.

A vector perpendicular to the xy-plane is parallel to the z-axis.

A direction vector is $(0, 0, 1)$.

Therefore, a vector equation of the line through $P(1, 2, 3)$ is $\vec{r} = (1, 2, 3) + t(0, 0, 1)$.

Vector Equation of a Line

Example

Determine whether the point $P(11, -8, 5)$ lies on the line $\vec{r} = (1, -2, 7) + t(5, -3, 1)$.

If the point is on the line, there must exist some value of t such that $(11, -8, 5) = (1, -2, 7) + t(5, -3, 1)$.

$$\begin{aligned} 11 &= 1 + 5t \qquad & -8 &= -2 - 3t \qquad & 5 &= 7 + t \\ t &= 2 & t &= 2 & t &= -2 \end{aligned}$$

Since we obtain different values for t, the point is not on the line.

Parametric Equation of a Line

In the previous example, we separated the x-, y- and z-components of the equation.

This extends the parametric form used in two-space to its three-dimensional equivalent.

Parametric Equations of a Line In Three-Space

The parametric equations of a line in three-space are $x = x_p + x_m t$, $y = y_p + y_m t$ and $z = z_p + z_m t$, where $\vec{p} = (x_p, y_p, z_p)$ is a position vector representing a point on the line, $\vec{m} = (x_m, y_m, z_m)$ is a direction vector for the line, and $t \in R$.

Parametric Equations of a Line

Example

Determine the parametric equations of the line that passes through $A(3, -5, -2)$ with direction vector $(-2, 3, -4)$.

The equations are $x = 3 - 2t$, $y = -5 + 3t$ and $z = -2 - 4t$.

Example

Determine the parametric equations of the line that passes through $A(1, 0, 4)$ that is parallel to the x-axis.

A direction vector is $\vec{m} = (1, 0, 0)$.

Therefore, the equations are $x = 1 + t$, $y = 0$ and $z = 4$.

Symmetric Equation of a Line

Like in two-space, it is possible to isolate the parameter t and create a symmetric equation without it.

Symmetric Equation of a Line In Three Space

The symmetric equation of a line in three-space is $\frac{x - x_p}{x_m} = \frac{y - y_p}{y_m} = \frac{z - z_p}{z_m}$, where $\vec{p} = (x_p, y_p, z_p)$ a position vector representing a point on the line, $\vec{m} = (x_m, y_m, z_m)$ is a direction vector for the line, and $x_m, y_m, z_m \neq 0$.

Symmetric Equation of a Line

Example

Determine the symmetric equation of the line that passes through $A(6, -1, 2)$ with direction vector $(4, 3, 5)$.

The equation is $\frac{x-6}{4} = \frac{y+1}{3} = \frac{z-2}{5}$.

Example

Determine the symmetric equation of the line that passes through $A(1, 0, 4)$ that is parallel to the line defined by $\vec{r} = (-3, -2, 4) + t(5, 2, 0)$.

Since the direction vector has a zero z-component, a symmetric equation is not possible.

A non-symmetric equation, however, is $\frac{x-1}{5} = \frac{y}{2}, z = 4$.

Symmetric Equation of a Line

Example

Represent the line passing through $P(1, 0, -3)$ and $Q(2, 1, 0)$ using vector, parametric and symmetric forms.

A direction vector is $\vec{m} = \vec{PQ} = (1, 1, 3)$.

Using point P, a vector equation is $\vec{r} = (1, 0, -3) + t(1, 1, 3)$.

The parametric equations are $x = 1 + t$, $y = t$ and $z = -3 + 3t$.

The symmetric equation is $x - 1 = y = \frac{z+3}{3}$.

Questions?