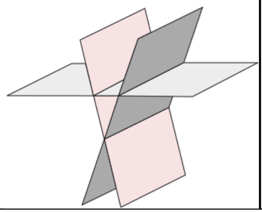
MCV4U: Calculus & Vectors

Solving Linear Systems Using Matrices

J. Garvin



Solving Linear Systems Using Matrices

Consider the system of equations below.

$$\begin{aligned} x + y + 2z &= 6 \qquad (1)\\ 3x - y - 3z &= 7 \qquad (2)\\ 2x + 2y + 7z &= 9 \qquad (3)\end{aligned}$$

Using elimination, we can determine that the solution to the system is $(3, 5, -1)$, which can be verified by substituting these values into all three equations.

Solving Linear Systems Using Matrices

An alternate method of solving systems of equations involves using *matrices*, which keep track of the coefficients and constants (similar to synthetic division of polynomials).

The same system of equations could be represented using the matrix below, where the first three columns record the coefficients of the x, y and z terms, and the final column records the constants.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6\\ 3 & -1 & -3 & 7\\ 2 & 2 & 7 & 9\end{array}\right]$$

The leftmost three columns form a *coefficient matrix*. The rightmost column is a *constant matrix*. Together, they form an *augmented matrix*.

Solving Linear Systems Using Matrices

A matrix is in *echelon form* if all rows below the downward-sloping diagonal contain zeroes, as in the following matrix.

$$\left[\begin{array}{ccc|c} a & b & c & d\\ 0 & e & f & g\\ 0 & 0 & h & i\end{array}\right]$$

If a matrix contains only 1s on this diagonal, and zeroes everywhere else (except possibly the final column), it is in *row-reduced echelon form* (RREF).

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a\\ 0 & 1 & 0 & b\\ 0 & 0 & 1 & c\end{array}\right]$$

RREF matrices may have zeroes on the diagonal, provided that leading zeroes are further right on subsequent rows.

Solving Linear Systems Using Matrices

Matrices in RREF are useful, because they represent the solutions to a system of equations.

For example, the matrix below indicates that the solution to some system of equations is $x = 5$, $y = -2$ and $z = 4$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & 4\end{array}\right]$$

Thus, if we are able to convert a matrix to RREF, we can obtain these solutions.

Solving Linear Systems Using Matrices

There are three main rules that can be used to convert a standard matrix to RREF.

Elementary Row Operations to Convert to RREF

1. Any two rows can swap positions.
2. Any row can be replaced by a scalar multiple of itself.
3. Two rows (or scalar multiples of them) can be added or subtracted, in order to replace one of the two rows.

Often, these steps are combined to reduce the amount of writing.

Solving Linear Systems Using Matrices

Example

Solve the linear system

$$\begin{aligned} x + y + 2z &= 6 \quad (1) \\ 3x - y - 3z &= 7 \quad (2) \\ 2x + 2y + 7z &= 9 \quad (3) \end{aligned}$$

First, convert the linear system to an augmented matrix, as we did earlier.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 3 & -1 & -3 & 7 \\ 2 & 2 & 7 & 9 \end{array}\right]$$

Solving Linear Systems Using Matrices

Next, use elementary row operations to force leading zeroes in rows 2 and 3 (R2 and R3). Specifically, we replace R2 with R2-3(R1), and R3 with R3-2(R1).

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 3 & -1 & -3 & 7 \\ 2 & 2 & 7 & 9 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & -4 & -9 & -11 \\ 0 & 0 & 3 & -3 \end{array}\right]$$

The matrix is in echelon form, but has not yet been reduced. We can multiply R3 by $\frac{1}{3}$ to solve for z.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & -4 & -9 & -11 \\ 0 & 0 & 3 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & -4 & -9 & -11 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Therefore, $z = -1$ is part of the solution.

Solving Linear Systems Using Matrices

We still need to convert the 1 and 2 in R1, and the -9 in R2, to zeroes to express the matrix in RREF. Start with the entry in R1C2 by replacing R1 with 4(R1)+R2.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & -4 & -9 & -11 \\ 0 & 0 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 4 & 0 & -1 & 13 \\ 0 & -4 & -9 & -11 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Now "zero-out" the -1 and -9 in C3 by replacing R1 with R1+R3, and R2 with R2+9(R3).

$$\left[\begin{array}{ccc|c} 4 & 0 & -1 & 13 \\ 0 & -4 & -9 & -11 \\ 0 & 0 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 4 & 0 & 0 & 12 \\ 0 & -4 & 0 & -20 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Solving Linear Systems Using Matrices

The matrix is nearly in RREF. All that remains is to multiply R1 by $\frac{1}{4}$ and R2 by $-\frac{1}{4}$.

$$\left[\begin{array}{ccc|c} 4 & 0 & 0 & 12 \\ 0 & -4 & 0 & -20 \\ 0 & 0 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Therefore, we can confirm that the solution to the system of equations is $x = 3$, $y = 5$ and $z = -1$, or $(3, 5, -1)$.

Solving Linear Systems Using Matrices

Example

Solve the linear system

$$\begin{aligned} 2y + 3z &= 7 \quad (1) \\ x + 2y - 4z &= -1 \quad (2) \\ 5x - 2y + 2z &= -7 \quad (3) \end{aligned}$$

In an augmented matrix form, the system of equations is

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 7 \\ 1 & 2 & -4 & -1 \\ 5 & -2 & 2 & -7 \end{array}\right]$$

Solving Linear Systems Using Matrices

Note that the first equation has a coefficient of 0 for x. We can swap R1 and R2, so that the leading zero appears in the second row instead.

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 7 \\ 1 & 2 & -4 & -1 \\ 5 & -2 & 2 & -7 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 2 & 3 & 7 \\ 5 & -2 & 2 & -7 \end{array}\right]$$

Zero out the 5 in R3 by replacing R3 with R3-5(R1).

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 2 & 3 & 7 \\ 5 & -2 & 2 & -7 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 2 & 3 & 7 \\ 0 & -12 & 22 & -2 \end{array}\right]$$

Solving Linear Systems Using Matrices

Zero out the two entries in column 2 by replacing R1 with R1-R2, and R3 with R3+6(R2).

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 2 & 3 & 7 \\ 0 & -12 & 22 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -7 & -8 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 40 & 40 \end{array}\right]$$

Simplify R3 by multiplying by $\frac{1}{40}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & -8 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 40 & 40 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -7 & -8 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, $z = 1$.

Solving Linear Systems Using Matrices

Zero out the two entries in column 3 by replacing R1 with R1+7(R3), and R2 with R2-3(R3).

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & -8 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Multiply R2 by $\frac{1}{2}$ to put it in RREF.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, the solution to the system is $(-1, 2, 1)$.

Solving Linear Systems Using Matrices

Example

Show that the linear system below is inconsistent.

$$\begin{aligned} x + y + z &= 5 \quad (1) \\ 1x - 2y - 4z &= 8 \quad (2) \\ x + 7y + 11z &= 3 \quad (3) \end{aligned}$$

As always, convert the linear system to an augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & -2 & -4 & 8 \\ 1 & 7 & 11 & 3 \end{array}\right]$$

Solving Linear Systems Using Matrices

Start by replacing R2 with R1-R2, and R3 with R1-R3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & -2 & -4 & 8 \\ 1 & 7 & 11 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 3 & 5 & -3 \\ 0 & -6 & -10 & 2 \end{array}\right]$$

Reduce R3 by multiplying by $-\frac{1}{2}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 3 & 5 & -3 \\ 0 & -6 & -10 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 3 & 5 & -3 \\ 0 & 3 & 5 & -1 \end{array}\right]$$

Since R2 and R3 have the same coefficients, but different constants, the system can have no solutions and is inconsistent.

Solving Linear Systems Using Matrices

To demonstrate the inconsistency another way, replace R3 with R3-R2.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 3 & 5 & -3 \\ 0 & 3 & 5 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 3 & 5 & -3 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

The last line states that $0x + 0y + 0z = 2$, which is impossible for any real values of x, y and z. Therefore, the system of equations is inconsistent.

Solving Linear Systems Using Matrices

Example

Show that the linear system below has infinitely many solutions.

$$\begin{aligned} x + y + z &= 5 \quad (1) \\ 1x + 3y - 5z &= 1 \quad (2) \\ x - 2y + 10z &= 11 \quad (3) \end{aligned}$$

Again, convert the linear system to an augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & -5 & 1 \\ 1 & -2 & 10 & 11 \end{array}\right]$$

Solving Linear Systems Using Matrices

Replace R2 with R1-R2, and R3 with R1-R3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & -5 & 1 \\ 1 & -2 & 10 & 11 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -2 & 6 & 4 \\ 0 & 3 & -9 & -6 \end{array}\right]$$

Multiply R2 by $-\frac{1}{2}$ and R3 by $\frac{1}{3}$ to reduce.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -2 & 6 & 4 \\ 0 & 3 & -9 & -6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & -2 \end{array}\right]$$

Since R2 and R3 are the same, it will be impossible for there to be a unique solution to the system of equations.

Solving Linear Systems Using Matrices

To better illustrate the situation, replace R3 with R2-R3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

R1 and R2 do not contain contradictory equations, and it is not possible to use the values in the two rows to uniquely solve for x and y. Therefore, there are an infinite number of solutions.

Questions?