MCV4U: Calculus & Vectors

Limits of Functions

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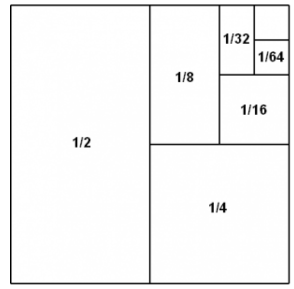
Limits

What is the value of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$?

A visual depiction is below.



Limits

If we use the first four terms of the sequence, then $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$.

If we increase the number of terms, we obtain the following:

Terms	5	6	7	...	20
Sum	$\frac{31}{32}$	$\frac{63}{64}$	$\frac{127}{128}$	...	$\frac{1048575}{1048576}$

As the number of terms increases, the sum approaches 1. We call this concept a *limit*.

Limits

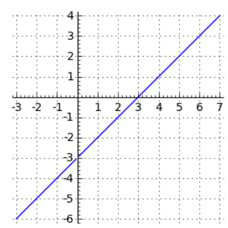
A limit is some value that a function (or sequence) approaches, as the input (or index) approaches some value.

Limits

Example

Determine the value of $\lim_{x \to 4} (x - 3)$.

This is a linear function, $f(x) = x - 3$, whose graph is below.



Limits

The expression $\lim_{x \to 4} (x - 3)$ means "what value does the linear function approach as x gets closer to 4?"

By observation, as $x \to 4$, $f(x) \to 1$.

Therefore, we state that $\lim_{x \to 4} (x - 3) = 1$.

In this example, it is also true that $f(4) = 1$, but this does not always need to be true.

Limits

Example

Determine the value of $\lim_{x \to \infty} \frac{1}{x}$.

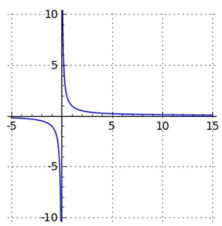
This time, we are not approaching a specific *value*, but ∞ itself.

Recall that the end behaviour of the function $\frac{1}{x}$ is defined by values of x that approach ∞.

Thus, the question can be restated as "does the end behaviour of $f(x) = \frac{1}{x}$ cause it to approach a specific value?"

Again, a graph of the function (or a knowledge of its basic properties) is useful.

Limits



The graph of $f(x) = \frac{1}{x}$ has a horizontal asymptote at $f(x) = 0$, and as $x \to \infty$, $f(x) \to 0$.

While the function never actually takes on a value of 0, it gets *infinitesimally close* to 0 and we say that $\lim\limits_{x \to \infty} \frac{1}{x} = 0$.

Limits

Example

Determine the value of $\lim\limits_{x \to 0} \frac{x-3}{x^2}$.

While it is possible to graph this rational function, an alternative method is to use a table of values that become closer and closer to 0.

First, check values that are less than 0.

x	-0.1	-0.01	-0.001	...
$f(x)$	-310	-30100	-3×10^6	...

$f(x)$ decreases rapidly, the closer it gets to 0. It appears that as $x \to 0$, $f(x) \to -\infty$.

Limits

Next, check values that are greater than 0.

x	...	0.001	0.01	0.1
$f(x)$	...	-3×10^6	-30100	-310

Again, $f(x)$ decreases rapidly and as $x \to 0$, $f(x) \to -\infty$.

Since all values suggest that $f(x)$ continues to decrease the closer it gets to 0, we say that $\lim\limits_{x \to 0} \frac{x-3}{x^2} = -\infty$.

Remember that ∞ is not a value. A function can approach ∞, but will never reach it!

One-Sided Limits

In the last example, we tested values close to a specific value.

A limit that approaches a certain value "from the left" or "from the right" is called a *one-sided limit*.

Left- and Right-Handed Limits

For a function $f(x)$, we denote as follows:

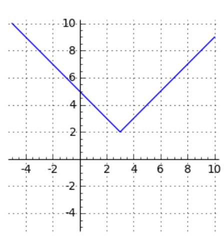
- the limit as x approaches a from the left, $\lim\limits_{x \to a^-} f(x)$, is called the *left-handed limit*
- the limit as x approaches a from the right, $\lim\limits_{x \to a^+} f(x)$, is called the *right-handed limit*

The values of the left- and right-handed limits may be different, depending on the function, or they may not exist at all.

One-Sided Limits

Example

For the function below, state the values of $\lim\limits_{x \to 3^-} f(x)$ and $\lim\limits_{x \to 3^+} f(x)$, if they exist.



One-Sided Limits

Moving from the left, $\lim\limits_{x \to 3^-} f(x) = 2$.

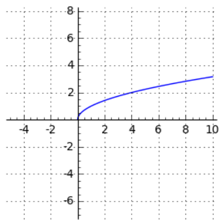
Moving from the right, $\lim\limits_{x \to 3^+} f(x) = 2$.

In this case, the left- and right-handed limits are equal. This is not always true.

One-Sided Limits

Example

For the function below, state the values of $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$, if they exist.



One-Sided Limits

Moving from the right, $\lim_{x \to 0^+} f(x) = 0$.

It is not possible to approach 0 from the left, however, since $f(x)$ is not defined for any $x < 0$.

Therefore, $\lim_{x \to 0^-} f(x)$ does not exist.

Limit of a Function

We can define the limit of a function using left- and right-handed limits.

Limit of a Function

Given a function $f(x)$, the limit as $x \to a$ exists if the left- and right-handed limits exist and are equal. Mathematically, $\lim_{x \to a} f(x) = L$ if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

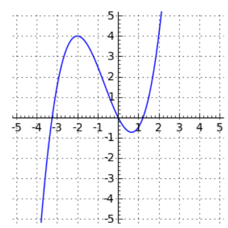
Using this definition, the limit of $f(x)$ as $x \to 0$ in the previous example does not exist, since $\lim_{x \to 0^-} f(x)$ does not exist.

A more formal definition of limits involving small quantities δ and ϵ is typically covered in first-year university courses, but this definition will suit us for now.

Limit of a Function

Example

Determine $\lim_{x \to 2} f(x)$ for $f(x)$ below, if it exists.

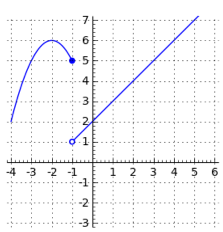


Since $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 4$, then $\lim_{x \to 2} f(x) = 4$.

Limit of a Function

Example

Determine $\lim_{x \to -1} f(x)$ for $f(x)$ below, if it exists.

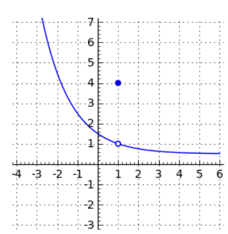


Since $\lim_{x \to -1^-} f(x) = 5$ and $\lim_{x \to -1^+} f(x) = 1$, then $\lim_{x \to -1} f(x)$ does not exist.

Limit of a Function

Example

Determine $\lim_{x \to 1} f(x)$ for $f(x)$ below, if it exists.



Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 1$, then $\lim_{x \to 1} f(x) = 1$.

Limit of a Function

Note that $f(x)$ is *discontinuous* at $x = 1$, and that $f(1) = 4$. We will talk about this in more detail soon.

Limits describe what is happening *around* a particular value. There is no requirement that the limit of the function as x approaches a is the same value as $f(a)$.

Questions?