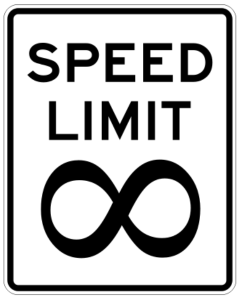
MCV4U: Calculus & Vectors

The Derivative of a Function

J. Garvin



Slope of a Tangent

Recap

Determine the slope of the tangent to $f(x) = 5x^2$ when $x = -2$.

Use the difference quotient with $x = -2$ and $f(-2) = 20$.

$$\begin{aligned} m_{\text{tangent}} &= \frac{5(-2+h)^2 - 20}{h} \\ &= \frac{5(4-4h+h^2) - 20}{h} \\ &= \frac{-20h + 5h^2}{h} \\ &= -20 + 5h \end{aligned}$$

As $h \to 0$, $-20 + 5h \to -20$. Therefore, the slope of the tangent to $f(x) = 5x^2$ at $x = -2$ is -20.

The Derivative of a Function

In the previous example, we let $h \to 0$ to create an infinitesimally small interval for our secant, essentially creating a tangent at $x = -2$.

We recognize this as the *limit* of the difference quotient as $h \to 0$.

This gives us a more formal definition using limits.

Limit Definition of the Derivative

For a function $y = f(x)$, the *derivative* may be expressed as:

- $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ in Lagrange notation, or
- $\dfrac{dy}{dx} = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ in Leibniz notation

The Derivative of a Function

The derivative is an expression that can be used to evaluate the instantaneous rate of change at a given point on a curve, or the slope of the tangent at that point.

The process of determining the derivative of a function is called *differentiation*.

If the derivative can be found at a given point, then the function is *differentiable* at that point. It is not always possible to determine the derivative of a function for all points on its domain.

Using this limit definition of the derivative is sometimes referred to as *differentiating using first principles*.

The Derivative of a Function

In all previous examples, we have substituted in a value for x to create an expression that is specific to that value.

Doing this has limited use, since if we need to find the instantaneous rate of change at another point, we must repeat the process using the new value.

Instead, we will create a *general expression* into which we can substitute any number of values for x. This requires very little extra effort on our part.

The Derivative of a Function

Example

Determine the derivative of $f(x) = 3x^2 + 5$ at $x = 2$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{[3(x+h)^2 + 5] - [3x^2 + 5]}{h} \\ &= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) + 5 - 3x^2 - 5}{h} \\ &= \lim_{h \to 0} \frac{6xh + 3h^2}{h} \\ &= \lim_{h \to 0} \frac{h(6x + 3h)}{h} \\ &= \lim_{h \to 0} (6x + 3h) \\ &= 6x \end{aligned}$$

The Derivative of a Function

Now that we have a general expression for $f'(x)$, we can use it to evaluate the rate of change when $x = 2$.

$$\begin{aligned} f'(2) &= 6(2) \\ &= 12 \end{aligned}$$

If we needed to evaluate the instantaneous rate of change at other points on the curve, say at $x = 10$, the second method tells us that $f'(10) = 6(10) = 60$.

Note that the second calculation was very fast!

The Derivative of a Function

Example

Determine the instantaneous rates of change of $y = \sqrt{x}$ when $x = 4$ and $x = 9$.

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \end{aligned}$$

The Derivative of a Function

As $h \to 0$, the denominator cleans up nicely.

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

When $x = 4$, $\left.\frac{dy}{dx}\right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$.

When $x = 9$, $\left.\frac{dy}{dx}\right|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$.

The Derivative of a Function

Example

Determine the equation of the tangent to $f(x) = \frac{1}{x}$ at $x = 5$.

The slope of the tangent at $x = 5$ is given by $f'(5)$.

$$\begin{aligned} f'(5) &= \lim_{h \to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h} \\ &= \lim_{h \to 0} \frac{5 - (5+h)}{5h(5+h)} \\ &= \lim_{h \to 0} \frac{-1}{5(5+h)} \\ &= -\tfrac{1}{25} \end{aligned}$$

The Derivative of a Function

When $x = 5$, $f(5) = \frac{1}{5}$, so the point of tangency is $\left(5, \frac{1}{5}\right)$.

Use the slope and point to find the equation of the tangent.

$$\begin{aligned} y &= -\tfrac{1}{25}(x - 5) + \tfrac{1}{5} \\ &= -\tfrac{1}{25}x + \tfrac{1}{5} + \tfrac{1}{5} \\ &= -\tfrac{1}{25}x + \tfrac{2}{5} \end{aligned}$$

Therefore, the tangent to $f(x) = \frac{1}{x}$ at $x = 5$ has the equation $y = -\frac{1}{25}x + \frac{2}{5}$.

Questions?