MCV4U: Calculus & Vectors

Intersections of Two Planes

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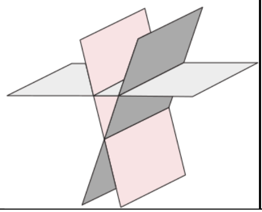
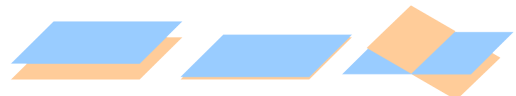


Intersections of Two Planes

There are three ways in which two planes may intersect each other (or not).

- Both planes are parallel and distinct (inconsistent)
- Both planes are coincident (infinite solutions)
- The two planes intersect in a line (infinite solutions)



The first two cases can be checked by examining the normals. The third case is more interesting.

Intersections of Two Planes

Example

Determine if the planes $\pi_1 : 3x - 2y + z - 7 = 0$ and $\pi_2 : 6x - 4y + 2z - 9 = 0$ intersect.

The two normals are $\vec{n_1} = (3, -2, 1)$ and $\vec{n_2} = (6, -4, 2)$.

Since $\vec{n_2} = 2\vec{n_1}$, but the equation for π_2 is not twice that of π_1, the two planes are parallel and distinct.

Therefore, there are no points of intersection.

Intersections of Two Planes

Example

Determine any points of intersection of the planes $\pi_1 : 4x - 4y - 2z - 10 = 0$ and $\pi_2 : \vec{r} = (3, 1, -1) + s(1, 0, 2) + t(1, 1, 0)$.

The normal for π_1 is $(4, -4, -2)$, and the normal for π_2 is $\vec{n_2} = (1, 0, 2) \times (1, 1, 0) = (-2, 2, 1)$.

Since $\vec{n_1} = -2\vec{n_2}$, the planes are either parallel and distinct or coincident.

Testing point $(3, 1, -1)$ in the equation for π_1 gives $4(3) - 4(1) - 2(-1) - 10 = 0$, so the point is common to both planes.

Therefore, the planes are coincident and there are an infinite number of intersections.

Intersections of Two Planes

Example

Determine parametric equations for the line of intersection of the planes $\pi_1 : 2x - 2y + 5z + 10 = 0$ and $\pi_2 : 2x + y - 4z + 7 = 0$.

The planes are not parallel, since $\vec{n_1} = (2, -2, 5)$ is not a scalar multiple of $\vec{n_2} = (2, 1, -4)$.

Since there are two equations with three variables, x, y, and z, the values of any two variables will be determined by the third.

The simplest method of obtaining parametric equations, then, is to assign an arbitrary parameter to one variable and base the other variables on it.

Intersections of Two Planes

Eliminate the x variable by subtracting the equation of π_1 from that of π_2.

$$\begin{aligned} 2x - 2y + 5z + 10 &= 0 \\ - \quad 2x + y - 4z + 7 &= 0 \\ -3y + 9z + 3 &= 0 \end{aligned}$$

Rearranging and simplifying this equation, $y = 3z + 1$.

Since y depends on z, assign a parameter to z.

For simplicity, let $z = t$. Then $y = 3t + 1$.

Using the equation for π_2, $2x + (3t + 1) - 4(t) + 7 = 0$, or $x = \frac{1}{2}t - 4$.

Thus, the line of intersection has parametric equations $x = \frac{1}{2}t - 4$, $y = 3t + 1$ and $z = t$.

Intersections of Two Planes

The choice of parameter is arbitrary, and can assign any value to any of the three variables.

For instance, if $z = 2t$ in the previous example, then the parametric equations of the line of intersection are $x = t - 4$, $y = 1 + 6t$ and $z = 2t$.

If $y = 3z - 1$ was rearranged for z instead of y, then $z = \frac{1}{3}y + \frac{1}{3}$.

Assigning $y = t$ gives the parametric equations $x = \frac{1}{6}t - \frac{17}{6}$, $y = t$, $z = \frac{1}{3}t + \frac{1}{3}$.

All of these represent the same line of intersection.

Intersections of Two Planes

Example

Represent the line $\vec{r} = (3, 0, 1) + t(1, -1, 2)$ as the intersection of two planes in scalar form.

Since the line is common to both planes, its direction vector can be used as a direction vector in each plane.

To create the first plane, construct a vector from the known point on the line to a point off of the line.

For example, the point $(0, 0, 0)$ is not on the line (you can verify this by trying to solve for t).

Therefore, an additional direction vector for the first plane is $(3 - 0, 0 - 0, 1 - 0) = (3, 0, 1)$.

Intersections of Two Planes

Find the normal to π_1 using the cross-product.

$$(1, -1, 2) \times (3, 0, 1) = (-1, 5, 3)$$

Find the equation of the plane by using the point $(3, 0, 1)$.

$$\begin{aligned} 1(3) - 5(0) - 3(1) + D &= 0 \\ D &= 0 \end{aligned}$$

Therefore, the equation of the plane is $x - 5y - 3z = 0$.

Intersections of Two Planes

To find an additional direction vector for π_2, we must be careful not to choose a direction vector that is parallel to that in π_1.

For instance, if we choose the direction vector $(2, -10, -6)$, then $2(3) - 10(0) - 6(1) + D = 0$, so $D = 0$. This means that the equation of π_2 is $2x - 10y - 6z = 0$, or $x - 5y - 3z = 0$.

Instead, using the point $(1, 0, 0)$, a direction vector for π_2 is $(3 - 1, 0 - 0, 1 - 0) = (2, 0, 1)$.

Since $(2, 0, 1)$ is not a scalar multiple of $(3, 0, 1)$, we may be able to use this vector to define π_2.

Intersections of Two Planes

It is possible, however, that this direction vector is *also* contained in π_1.

Recall that the triple-scalar-product was used to calculate the volume of a parallelepiped (i.e. it calculates the area of the base parallelogram and multiplies it by the height of the parallelepiped).

The TSP, then, can be used to check if three vectors are coplanar: if the TSP=0, there is no "height" and the vectors are coplanar.

Coplanar Vectors

Vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar if $\vec{u} \times \vec{v} \cdot \vec{w} = 0$.

Intersections of Two Planes

Calculate the TSP of $(1, -1, 2)$, $(3, 0, 1)$ and $(2, 0, 1)$, to see if they are coplanar.

$$\begin{aligned} (1, -1, 2) \times (3, 0, 1) \cdot (2, 0, 1) &= (-1, 5, 3) \cdot (2, 0, 1) \\ &= 1 \end{aligned}$$

Since the TSP is non-zero, the three vectors are not coplanar.

This means that $(2, 0, 1)$ is not contained in π_1, and we can use it as a basis for a second plane that contains $(1, -1, 2)$.

Intersections of Two Planes

Fine the normal for π_2.

$$(2,0,1) \times (1,-1,2) = (1,-3,-2)$$

Use $(3,0,1)$ and solve for D.

$$\begin{aligned} 1(3) - 3(0) - 2(1) + D &= 0 \\ D &= -1 \end{aligned}$$

Therefore an equation for π_2 is $x - 3y - 2z - 1 = 0$.

Therefore, the line can be represented as the intersection of the planes $\pi_1 : x - 5y - 3z = 0$ and $\pi_2 : x - 3y - 2z - 1 = 0$.

Questions?