MCV4U: Calculus & Vectors

Increasing/Decreasing Intervals on a Function

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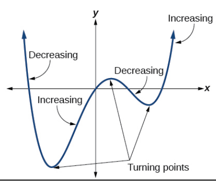
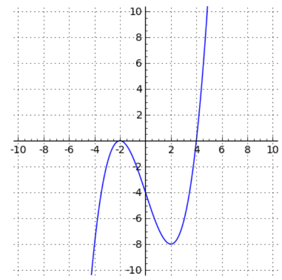


Increasing/Decreasing Intervals

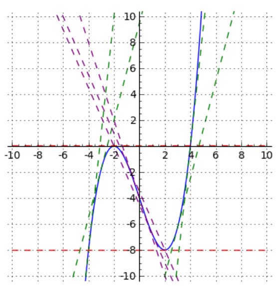
Consider the graph of $f(x) = \frac{1}{4}x^3 - 3x - 4$ below.



$f(x)$ is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$, decreasing on $(-2, 2)$, and "flat" when x is -2 or 2.

Increasing/Decreasing Intervals

Now, examine the tangents to $f(x)$.



Tangents with positive slopes are green, negative slopes are purple, and horizontal slopes are red.

Increasing/Decreasing Intervals

Strictly speaking, a function is *increasing* on an interval (a, b) if $f(q) > f(p)$ for all $a \leq p < q \leq b$.

Less formally, if we are able to select any two such values, p and q, and construct a secant between the points $(p, f(p))$ and $(q, f(q))$, then the slope of the secant should be positive if the function is increasing on (a, b).

If the interval spanned by the secant is infinitesimally small, then the slope of the secant is equivalent to the slope of the tangent at a given point.

This implies that the slope of the tangent to any point on an increasing interval should be positive.

Similarly, the slope of the tangent to any point on a *decreasing* interval should be negative.

Increasing/Decreasing Intervals

Test For Increasing/Decreasing Intervals

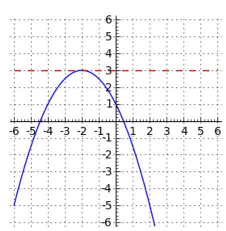
For any function $f(x)$, the function is:

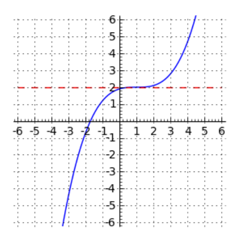
- increasing at x if $f'(x) > 0$,
- decreasing at x if $f'(x) < 0$, and
- "flat" at x if $f'(x) = 0$.

Note that if $f'(x) = 0$, we know that the tangent to the function is horizontal at that point, but we do not have enough information (yet) to determine if the function is increasing, decreasing or neither at x.

Increasing/Decreasing Intervals

The graph below left shows that $f'(-2) = 0$, with a maximum point at $(-2, 3)$, while the graph below right shows that $f'(1) = 0$, but the function is always increasing.





We will cover this in more detail next lesson.

Increasing/Decreasing Intervals

Example

Verify algebraically that the function $f(x) = \frac{1}{4}x^3 - 3x - 4$ is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$ and decreasing on $(-2, 2)$.

The derivative is $f'(x) = \frac{3}{4}x^2 - 3$.

To find the "flattening out" points, let $f'(x) = 0$ and solve for x.

$$\begin{aligned} \tfrac{3}{4}x^2 - 3 &= 0 \\ \tfrac{3}{4}x^2 &= 3 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Increasing/Decreasing Intervals

Therefore, $f(x)$ has horizontal tangents when $x = 2$ or $x = -2$.

This divides $f(x)$ into three intervals: $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$.

To determine whether $f(x)$ is increasing or decreasing on each interval, test any value within.

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
x	-3	0	3
$f'(x)$	15	-12	15
sign	$+$	$-$	$+$

Since $f'(x) > 0$ on the intervals $(-\infty, -2)$ and $(2, \infty)$, $f(x)$ is increasing on those intervals. Since $f'(x) < 0$ on $(-2, 2)$, $f(x)$ is decreasing there.

Increasing/Decreasing Intervals

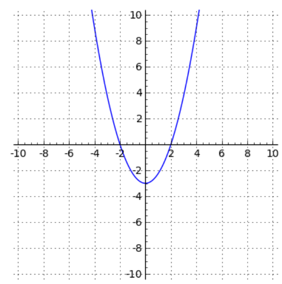
Recall that the derivative of a function is, itself, a function.

Therefore, if we make a graph of a function's derivative, all values above the x-axis represent positive rates of change (slopes), whereas those below the x-axis represent negative rates of change.

Given the graph of a function's derivative, then, we can determine the intervals of increase or decrease of the original function.

Increasing/Decreasing Intervals

The graph of $f'(x) = \frac{3}{4}x^2 - 3$ is below.



Note that the intervals $(-\infty, -2)$ and $(2, \infty)$ are positive, while the interval $(-2, 2)$ is negative.

Increasing/Decreasing Intervals

Example

State the intervals of increase/decrease for the function $f(x) = (x+1)^3(x-3)$.

Since $f(x)$ is a quartic function with a positive leading coefficient, it has Q2→Q1 end behaviour.

There is a repeated root with order 3 at $x = -1$ (passes through the x-axis similar to a cubic), and a single root at $x = 3$ (also passes through).

This suggests that there should be one minimum value, surrounded by a decreasing interval to the left and an increasing interval to the right.

Increasing/Decreasing Intervals

Use the product and chain rules to find the derivative.

$$\begin{aligned} f'(x) &= 3(x+1)^2(x-3) + (x+1)^3(1) \\ &= (x+1)^2(3(x-3) + (x+1)) \\ &= (x+1)^2(4x-8) \\ &= 4(x+1)^2(x-2) \end{aligned}$$

Note that it would have been possible to determine the derivative by expanding $f(x)$, using the power rule, and then using the factor theorem to simplify $f'(x)$, but this would involve more work than simply using the product and chain rules and common factoring.

Increasing/Decreasing Intervals

Since $f'(x)$ has zeroes at -1 and 2, divide it into the intervals $(-\infty, -1)$, $(-1, 2)$ and $(2, \infty)$, then test values in each interval.

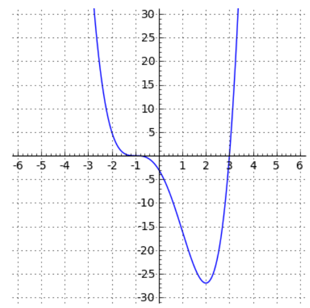
Interval	$(-\infty, -1)$	$(-1, 2)$	$(2, \infty)$
x	-2	0	3
$f'(x)$	-16	-8	64
sign	$-$	$-$	$+$

Since $f'(x) < 0$ for both intervals $(-\infty, -1)$ and $(-1, 2)$, $f(x)$ is decreasing on the interval $(-\infty, 2)$.

Since $f'(x) > 0$ on the interval $(2, \infty)$, $f(x)$ is increasing on that interval.

Increasing/Decreasing Intervals

A graph of $f(x) = (x+1)^3(x-3)$ confirms these intervals.



Increasing/Decreasing Intervals

Example

Show that the function $y = -\frac{1}{5}x^5 + x^4 + 4x^3 - 16x^2 - 64x$ is continuously decreasing.

First, find the derivative using fundamental rules.

$$\frac{dy}{dx} = -x^4 + 4x^3 + 12x^2 - 32x - 64$$

Use the factor theorem to determine factors of y. In this case, $y = 0$ when $x = 4$ and when $x = -2$, so $x - 4$ and $x + 2$ are factors of y.

A combination of polynomial division (long or synthetic) results in a quadratic factor that itself can be factored.

Increasing/Decreasing Intervals

After factoring, we obtain $\frac{dy}{dx} = -(x-4)^2(x+2)^2$.

We need to check the intervals $(-\infty, -2)$, $(-2, 4)$ and $(4, \infty)$.

It is not necessary to determine the actual *values* of the derivative in each interval, only their signs.

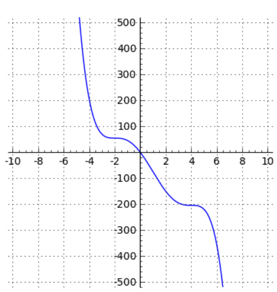
In this case, any values of x will result in non-negative values, due to the two squares.

The leading negative will, therefore, make all values of the derivative either zero (at $x = 4$ and $x = -2$) or negative.

Therefore, the function is continuously decreasing.

Increasing/Decreasing Intervals

A graph (zoomed in) shows that the function is always decreasing.



Questions?