MCV4U: Calculus & Vectors

Higher-Order Derivatives

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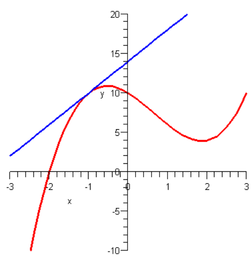


Implicit Differentiation

Recap

Determine the derivative of $y^3 + 4x^2y^5 - 5x = 10$.

Remember that all terms involving y will have $\frac{dy}{dx}$ in their derivations, and that the central term uses the product rule.

$$\begin{aligned} \tfrac{d}{dx}\left(y^3 + 4x^2y^5 - 5x\right) &= \tfrac{d}{dx}10 \\ \tfrac{d}{dx}y^3 + 4\tfrac{d}{dx}x^2y^5 - 5\tfrac{d}{dx}x &= 0 \\ 3y^2\tfrac{dy}{dx} + 4\left(2xy^5 + x^2(5y^4)\tfrac{dy}{dx}\right) - 5 &= 0 \\ 3y^2\tfrac{dy}{dx} + 8xy^5 + 20x^2y^4\tfrac{dy}{dx} - 5 &= 0 \\ \tfrac{dy}{dx} &= \frac{5 - 8xy^5}{3y^2 + 20x^2y^4} \end{aligned}$$

Higher-Order Derivatives

Recall that the derivative of a function represents its rate of change. This is often called the *first derivative*.

What if we are interested in the rate of change of the derivative itself – that is, how is the derivative changing with respect to the independent variable?

This concept is known as the *second derivative*.

The Second Derivative

The second derivative is the derivative of the derivative function. In Lagrange notation, the it is denoted $f''(x)$. In Leibniz notation, it is denoted $\frac{d^2y}{dx^2}$.

To find the second derivative, differentiate the first derivative.

Higher-Order Derivatives

Example

Determine the second derivative of $f(x) = -5x^3 + 4x^2$.

Using the power rule, $f'(x) = -15x^2 + 8x$.

Differentiating the first derivative, $f''(x) = -30x + 8$.

Example

Determine the second derivative of $f(x) = \sqrt{x} + 3$.

Using $f(x) = x^{\frac{1}{2}} + 3$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$.

Differentiating this, $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$.

Higher-Order Derivatives

Example

If $y = \sqrt{3x^2 + 5}$, determine $\frac{d^2y}{dx^2}$.

Use the chain rule on $y = \left(3x^2 + 5\right)^{\frac{1}{2}}$ to find $\frac{dy}{dx}$.

$$\begin{aligned} \tfrac{dy}{dx} &= \tfrac{1}{2}(3x^2 + 5)^{-\frac{1}{2}}(6x) \\ &= \frac{3x}{(3x^2 + 5)^{\frac{1}{2}}} \end{aligned}$$

Use the quotient rule (or product and chain rules) to find $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{3(3x^2 + 5)^{\frac{1}{2}} - 3x\left(\frac{1}{2}\right)(3x^2 + 5)^{-\frac{1}{2}}(6x)}{\left[(3x^2 + 5)^{\frac{1}{2}}\right]^2}$$

which simplifies to $\dfrac{15}{(3x^2 + 5)^{\frac{3}{2}}}$ after quite a bit of algebra.

Higher-Order Derivatives

Third, fourth or even higher-order derivatives can be calculated by further differentiation.

Higher-Order Derivatives

In Lagrange notation, the nth derivative is denoted $f^{(n)}(x)$. In Leibniz notation, it is denoted $\frac{d^ny}{dx^n}$.

The third derivative can be either $f'''(x)$ or $f^{(3)}(x)$ in Lagrange notation, and $\frac{d^3y}{dx^3}$ in Leibniz notation.

After the third derivative, prime notation becomes cumbersome for Lagrange notation, so numbers are used instead.

For instance, both $\frac{d^9y}{dx^9}$ and $f^{(9)}(x)$ are more readable than $f'''''''''(x)$.

Higher-Order Derivatives

Example

Determine the third derivative of $f(x) = 6x^5 - 3x^2 + 5x$.

Using the power rule, the first derivative is given by $f'(x) = 30x^4 - 6x + 5$.

The second derivative is the derivative of $f'(x)$, so $f''(x) = 120x^3 - 6$.

The third derivative is the derivative of $f''(x)$, so $f'''(x) = 360x^2$.

Higher-Order Derivatives

Example

If $y = 2x^{99}$, determine $\frac{d^{100}y}{dx^{100}}$.

Recall that the power rule decrements the exponent by 1 each time it is applied.

For instance, the derivative of the quadratic function $y = x^2$ is a linear function $\frac{dy}{dx} = 2x$.

The second derivative is $\frac{d^2y}{dx^2} = 2$, a constant, and the third derivative is $\frac{d^3y}{dx^3} = 0$.

In general, a polynomial function of degree k will have a nth derivative of degree $k - n$.

Since $99 - 100 < 0$, then $\frac{d^{100}y}{dx^{100}} = 0$.

Displacement, Velocity and Acceleration

Velocity is the change of an object's displacement with respect to time. Therefore, velocity is the derivative of the displacement function.

Velocity

If $s(t)$ represents an object's displacement with respect to time, then its velocity is given by $v(t) = s'(t)$. In Leibniz notation, $v = \frac{ds}{dt}$.

An object is moving in a positive direction if its velocity is positive, and in a negative direction if its velocity is negative.

An object with a velocity of zero is at rest.

Displacement, Velocity and Acceleration

Example

A ball is kicked off of a cliff and its height, h metres, after t seconds is given by $h(t) = -4.9t^2 + 19.6t + 24.5$. With what speed does the ball hit the ground?

The ball hits the ground when $h(t) = 0$.

$$-4.9t^2 + 19.6t + 24.5 = 0$$
$$-4.9(x + 1)(x - 5) = 0$$

The ball hits the ground at 5 s. Find $v(5)$ for its speed.

$$v(t) = -9.8t + 19.6$$
$$v(5) = -9.8(5) + 19.6$$
$$= -29.4$$

Thus, the ball hits the round with a speed of 29.4 m/s.

Displacement, Velocity and Acceleration

Acceleration is the change of an object's velocity with respect to time. Therefore, acceleration is the derivative of the velocity function.

Acceleration

If $s(t)$ represents an object's displacement with respect to time, and $v(t)$ its velocity, then its acceleration is given by $a(t) = v'(t)$ or $a(t) = d''(t)$. In Leibniz notation, $a = \frac{dv}{dt}$ or $a = \frac{d^2s}{dt^2}$.

An object with an acceleration of zero is moving at a constant velocity.

Note that it is possible for an object to have a negative acceleration, but a positive velocity – it may be moving forward, but slowing down.

Displacement, Velocity and Acceleration

Example

The displacement, in metres, of a particle after t seconds is given by $s(t) = -2t^3 + 8t^2 - 10t$.

- Determine the velocity of the particle at 2 seconds.
- When is the particle moving in a positive direction? In a negative direction? At rest?
- Determine the acceleration of the particle at 2 seconds.
- When is the particle moving with a constant velocity?

An expression for the velocity of the particle is $v(t) = s'(t) = -6t^2 + 16t - 10$.

At $t = 2$, $v(2) = -6(2)^2 + 16(2) - 10 = -2$ m/s.

Displacement, Velocity and Acceleration

Since $v(t)$ is a quadratic function whose graph opens downward, the particle will be moving in a positive direction for all values of t between the roots of the equation, and in a negative direction for all other values.

$$\begin{aligned} v(t) &= -6t^2 + 16t - 10 \\ &= -(t-1)(6t-10) \end{aligned}$$

Since the roots are 1 and $\frac{5}{3}$, the particle is moving in a positive direction between 1 and $\frac{5}{3}$ seconds.

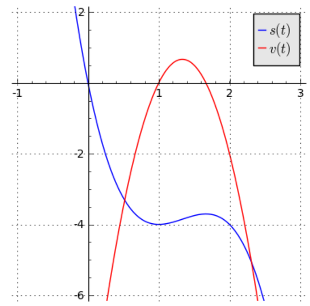
It is moving in a negative direction between 0 and 1 second, and for all times after $\frac{5}{3}$ seconds.

Therefore, the particle is moving in a positive direction before $\frac{5}{3}$ seconds, and in a negative direction after $\frac{5}{3}$ seconds.

At 1 second and at $\frac{5}{3}$ seconds the particle is briefly at rest.

Displacement, Velocity and Acceleration

A graph of the displacement, $s(t)$, and velocity, $v(t)$, is below.



Displacement, Velocity and Acceleration

An expression for the acceleration of the particle is $a(t) = v'(t) = -12t + 16$.

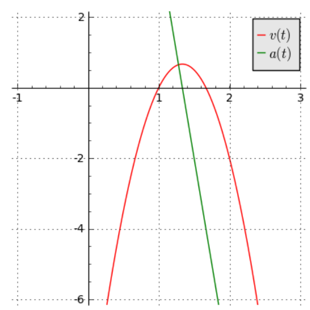
At $t = 2$ seconds, $a(2) = -12(2) + 16 = -8$ m/s^2.

An object moving at a constant velocity (possibly at rest) is not accelerating, so $a(t) = 0$.

Therefore, the particle is moving at a constant velocity when $-12t + 16 = 0$, or at $\frac{4}{3}$ seconds.

Displacement, Velocity and Acceleration

A graph of the velocity, $v(t)$, and acceleration, $a(t)$, is below.



Questions?