MCV4U: Calculus & Vectors

Applications of the Dot and Cross Products

Part 1: Geometric Applications

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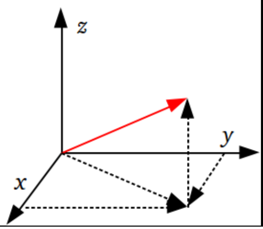
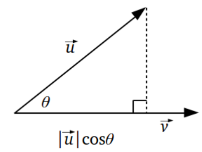


Projections

In the real world, a *projection* occurs when an object casts its shadow or image onto another object.

Mathematically, a projection of one vector onto another object (such as a vector, a line, or a plane) involves dropping a perpendicular line from the head of the vector to the object.



The magnitude of the projection (or *scalar projection*) of $\vec{u}$ onto $\vec{v}$ is $|\text{proj}_{\vec{v}}\vec{u}| = |\vec{u}|\cos\theta$ (sometimes denoted $|\vec{u} \downarrow \vec{v}|$).

Projections

By expressing a projection using the dot product, we can use algebraic vectors in our calculations.

$$\begin{aligned} |\text{proj}_{\vec{v}}\vec{u}| &= |\vec{u}|\cos\theta \times \frac{|\vec{v}|}{|\vec{v}|} \\ &= \frac{|\vec{u}||\vec{v}|\cos\theta}{|\vec{v}|} \\ &= \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} \end{aligned}$$

Scalar Projection of $\vec{u}$ Onto $\vec{v}$

The scalar projection of $\vec{u}$ onto $\vec{v}$ is $|\text{proj}_{\vec{v}}\vec{u}| = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$.

Projections

The scalar projection can be positive or negative, depending on the value of θ.

- If $0° < \theta < 90°$, then $\text{proj}_{\vec{v}}\vec{u} > 0$.
- If $90° < \theta < 180°$, then $\text{proj}_{\vec{v}}\vec{u} < 0$.
- If $\theta = 90°$, then $\text{proj}_{\vec{v}}\vec{u} = 0$.

Projections

The *vector projection* of $\vec{u}$ onto $\vec{v}$ is a vector with the same direction as $\vec{v}$ and a magnitude equal to the scalar projection.

Thus, multiplying the scalar projection with a unit vector in the direction of $\vec{v}$ produces the vector projection.

$$\begin{aligned} \text{proj}_{\vec{v}}\vec{u} &= \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} \times \frac{1}{|\vec{v}|}\vec{v} \\ &= \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} \end{aligned}$$

Vector Projection of $\vec{u}$ Onto $\vec{v}$

The vector projection of $\vec{u}$ onto $\vec{v}$ is $\text{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$.

Projections

Example

Calculate the scalar and vector projections of $\vec{a} = (3, 0, 5)$ onto $\vec{b} = (7, -1, 2)$.

$$\begin{aligned} |\text{proj}_{\vec{b}}\vec{a}| &= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} \\ &= \frac{3(7) + 0(-1) + 5(2)}{\sqrt{7^2 + (-1)^2 + 2^2}} \\ &= \frac{31}{\sqrt{54}} \text{ or } \frac{31\sqrt{54}}{54} \end{aligned}$$

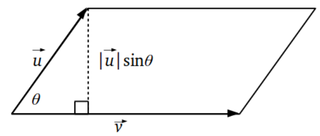
Projections

$$\begin{aligned}\text{proj}_{\vec{b}}\vec{a} &= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}\\ &= |\text{proj}_{\vec{b}}\vec{a}| \times \frac{\vec{b}}{|\vec{b}|}\\ &= \frac{31(7,-1,2)}{54}\\ &= \left(\tfrac{217}{54}, -\tfrac{31}{54}, \tfrac{31}{27}\right)\end{aligned}$$

Area of a Parallelogram

Recall that the magnitude of the cross product is given by $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$.

Consider the following diagram of a parallelogram.

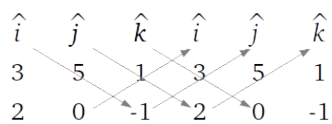


Since the area of a parallelogram is given by $A = bh$, the area of the parallelogram is the magnitude of the cross product.

Area of a Parallelogram

Example

Determine the area of a parallelogram with one vertex at the origin and two others at $(3,5,1)$ and $(2,0,-1)$.



$$\begin{aligned}\vec{u}\times\vec{v} &= (5(-1)-1(0), 1(2)-3(-1), 3(0)-5(2))\\ &= (-5,5,-10)\end{aligned}$$

Area of a Parallelogram

$$\begin{aligned}|\vec{u}\times\vec{v}| &= |(-5,5,-10)|\\ &= \sqrt{(-5)^2+5^2+(-10)^2}\\ &= 5\sqrt{6}\text{ units}^2\end{aligned}$$

Triple Scalar Product

A calculation involving both the dot and cross products is the *triple scalar product*.

Triple Scalar Product

For vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, the triple scalar product (TSP) is $\vec{u}\cdot\vec{v}\times\vec{w}$.

As its name suggests, the TSP produces a scalar value.

Since the dot and cross products both operate on vectors, the cross product *must* be performed first.

Triple Scalar Product

Example

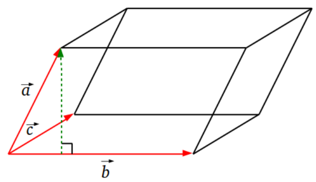
Determine the TSP of $\vec{p} = (1,2,0)$, $\vec{q} = (4,0,-3)$ and $\vec{r} = (3,-1,-2)$.

$$\begin{aligned}\vec{p}\cdot\vec{q}\times\vec{r} &= (1,2,0)\cdot(4,0,-3)\times(3,-1,-2)\\ &= (1,2,0)\cdot(-3,-1,-4)\\ &= -5\end{aligned}$$

Note that the TSP can be positive or negative.

Volume of a Parallelepiped

A *parallelepiped* is a six-faced, three-dimensional solid where opposite faces are parallel.Let $\vec{b}$ and $\vec{c}$ form the base of a parallelepiped, and $\vec{a}$ a non-coplanar edge, as shown.



Volume of a Parallelepiped

Like all prisms, the volume of a parallelepiped can be calculated as the product of the area of its base and its height.

Since the base is a parallelogram, the area of its base is $|\vec{b} \times \vec{c}|$.

The height of the parallelepiped is the magnitude of $\vec{a}$ projected onto the vector produced by $\vec{b} \times \vec{c}$.

The volume, v, of the parallelepiped is

$$\begin{aligned} V &= |\vec{b} \times \vec{c}| \frac{|\vec{a} \cdot \vec{b} \times \vec{c}|}{|\vec{b} \times \vec{c}|} \\ &= |\vec{a} \cdot \vec{b} \times \vec{c}| \end{aligned}$$

The volume of a parallelepiped is the magnitude of the TSP.

Volume of a Parallelepiped

Example

Determine the volume of the parallelepiped defined by $\vec{u} = (1, 0, 3)$, $\vec{v} = (4, 1, 0)$ and $\vec{w} = (2, -1, 1)$.

$$\begin{aligned} |\vec{u} \cdot \vec{v} \times \vec{w}| &= |(1, 0, 3) \cdot (4, 1, 0) \times (2, -1, 1)| \\ &= |(1, 0, 3) \cdot (1, -4, -6)| \\ &= 17 \text{ units}^3 \end{aligned}$$

Questions?