MCV4U: Calculus & Vectors

The Dot Product of Two Vectors

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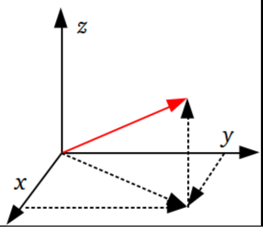
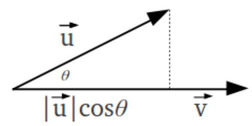


Geometric Definition of the Dot Product

We define the *dot product* of two geometric vectors, $\vec{u}$ and $\vec{v}$, as the product of their magnitudes multiplied by the cosine of the angle between them.

Dot Product of Two Geometric Vectors

For any two vectors $\vec{u}$ and $\vec{v}$, $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$, $0° \leq \theta \leq 180°$.



Geometric Definition of the Dot Product

Since $|\vec{u}|$, $|\vec{v}|$ and $\cos\theta$ are all scalars, the dot product is always a scalar quantity.

Because of this, the dot product is often referred to as the *scalar product*.

Depending on the measure of θ, the dot product can be positive, zero, or negative.

- if $0° \leq \theta < 90°$, $\vec{u} \cdot \vec{v} > 0$.
- if $\theta = 90°$, $\vec{u} \cdot \vec{v} = 0$.
- if $90° < \theta \leq 180°$, $\vec{u} \cdot \vec{v} < 0$.

Geometric Definition of the Dot Product

Example

Given $|\vec{a}| = 5$, $|\vec{b}| = 7$ and $\theta = 40°$, calculate $\vec{a} \cdot \vec{b}$ and $\vec{b} \cdot \vec{a}$.

$$\vec{a} \cdot \vec{b} = 5 \times 7 \times \cos 40° \approx 26.8$$
$$\vec{b} \cdot \vec{a} = 7 \times 5 \times \cos 40° \approx 26.8$$

Example

Using the same vectors, calculate $\vec{a} \cdot \vec{a}$.

$$\vec{a} \cdot \vec{a} = 5 \times 5 \times \cos 0° = 25$$

Geometric Definition of the Dot Product

In the previous example, $\vec{a} \cdot \vec{b}$ and $\vec{b} \cdot \vec{a}$ produced the same answer. This is because the dot product is *commutative*.

That is, for any vectors $\vec{u}$ and $\vec{v}$, $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

$$\begin{aligned}\vec{u} \cdot \vec{v} &= |\vec{u}||\vec{v}|\cos\theta \\ &= |\vec{v}||\vec{u}|\cos\theta \\ &= \vec{v} \cdot \vec{u}\end{aligned}$$

Geometric Definition of the Dot Product

In the previous example, we also saw that the dot product of a vector with itself results in the square of that vector's magnitude.

That is, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$.

This is because the angle between a vector and itself is $0°$, and $\cos 0° = 1$.

Geometric Definition of the Dot Product

Some other properties of the dot product:

- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributive property)
- $k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v} = \vec{u} \cdot (k\vec{v})$ (associative property)

Proofs of these are fairly easy to work out, and are left as exercises for the determined student.

Algebraic Definition of the Dot Product

We can also define the dot product for algebraic vectors.

Dot Product of Two Algebraic Vectors

For any two vectors $\vec{u} = (x_u, y_u, z_u)$ and $\vec{v} = (x_v, y_v, z_v)$, $\vec{u} \cdot \vec{v} = x_u x_v + y_u y_v + z_u z_v$.

$$\begin{aligned}\vec{u} \cdot \vec{v} =& (x_u\hat{i} + y_u\hat{j} + z_u\hat{k}) \cdot (x_v\hat{i} + y_v\hat{j} + z_v\hat{k}) \\ =& x_u x_v \hat{i}\hat{i} + x_u y_v \hat{i}\hat{j} + x_u z_v \hat{i}\hat{k} + \\ & y_u x_v \hat{i}\hat{j} + y_u y_v \hat{j}\hat{j} + y_u z_v \hat{j}\hat{k} + \\ & z_u x_v \hat{i}\hat{k} + z_u y_v \hat{j}\hat{k} + z_u z_v \hat{k}\hat{k}\end{aligned}$$

Algebraic Definition of the Dot Product

Recall that $\hat{a} \cdot \hat{a} = 1$ and $\hat{a} \cdot \hat{b} = 0$ when $\vec{a}$ and $\vec{b}$ are perpendicular.

$$\begin{aligned}\vec{u} \cdot \vec{v} =& x_u x_v(1) + x_u y_v(0) + x_u z_v(0) + \\ & y_u x_v(0) + y_u y_v(1) + y_u z_v(0) + \\ & z_u x_v(0) + z_u y_v(0) + z_u z_v(1) \\ =& x_u x_v + y_u y_v + z_u z_v\end{aligned}$$

Algebraic Definition of the Dot Product

Example

Calculate $\vec{u} \cdot \vec{v}$ given $\vec{u} = (4, -3, 5)$ and $\vec{v} = (1, 0, -2)$.

$$\begin{aligned}\vec{u} \cdot \vec{v} &= (4, -3, 5) \cdot (1, 0, -2) \\ &= 4 \cdot 1 + (-3) \cdot (0) + 5 \cdot (-2) \\ &= -6\end{aligned}$$

Algebraic Definition of the Dot Product

Example

Show that the triangle with vertices $A(3, -4)$, $B(1, 2)$ and $C(-8, -1)$ contains a right angle.

If there is a right angle, the dot product of two vectors will be zero. In this case, the right angle is between $\vec{AB}$ and $\vec{BC}$.

$$\begin{aligned}\vec{AB} \cdot \vec{BC} &= (1 - 3, 2 - (-4)) \cdot (-8 - 1, -1 - 2) \\ &= (-2, 6) \cdot (-9, -3) \\ &= (-2) \cdot (-9) + 6 \cdot (-3) \\ &= 0\end{aligned}$$

Algebraic Definition of the Dot Product

Example

For vectors $\vec{a} = (3, 6, -4)$ and $\vec{b} = (-2, k, 1)$, determine the value of k such that the two vectors are perpendicular.

$$\begin{aligned}\vec{a} \cdot \vec{b} &= 0 \\ 3 \cdot (-2) + 6 \cdot k - 4 \cdot 1 &= 0 \\ -6 + 6k - 4 &= 0 \\ 6k &= 10 \\ k &= \frac{5}{3}\end{aligned}$$

Algebraic Definition of the Dot Product

Example

Determine the angle between $\vec{u} = (5, -1, 2)$ and $\vec{v} = (4, 0, 3)$.

$$\begin{aligned}\vec{u} \cdot \vec{v} &= 5 \cdot 4 - 1 \cdot 0 + 2 \cdot 3 \\ &= 26 \\ |\vec{u}| &= \sqrt{5^2 + (-1)^2 + 2^2} \\ &= \sqrt{30} \\ |\vec{v}| &= \sqrt{4^2 + 0^2 + 3^2} \\ &= 5 \\ \theta &= \cos^{-1}\left(\frac{26}{5\sqrt{30}}\right) \\ &\approx 18^\circ\end{aligned}$$

Questions?