MCV4U: Calculus & Vectors

Derivative of the Natural Logarithmic Function, $y = \ln x$

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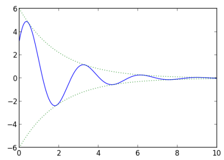

Derivatives Involving e^x

Recap

Determine the derivative of $f(x) = \frac{4x^3}{e^{5x}}$.

$$\begin{aligned} f'(x) &= \frac{e^{5x} \cdot 12x^2 - 4x^3 \cdot 5e^{5x}}{[e^{5x}]^2} \\ &= \frac{e^{5x}(12x^2 - 20x^3)}{[e^{5x}]^2} \\ &= \frac{12x^2 - 20x^3}{e^{5x}} \\ &= -\frac{4x^2(5x-3)}{e^{5x}} \end{aligned}$$

Derivative of $y = \ln x$

Many phenomena are modelled using logarithmic functions.

A general logarithmic function, $y = \log_b x$, is related to an exponential function.

$$\text{If } y = \log_b x \text{ then } b^y = x$$

A logarithmic function with a base of e, $y = \log_e x$, is often abbreviated $y = \ln x$ and is called the "natural logarithm".

Note that $\ln e^x = \log_e e^x = x$, and $e^{\ln x} = e^{\log_e x} = x$.

Derivative of $y = \ln x$

To find the derivative of $y = \ln x$, rewrite using base e on both sides of the equation and implicitly differentiate.

$$\begin{aligned} y &= \ln x \\ e^y &= e^{\ln x} \\ e^y &= x \\ \tfrac{d}{dx} e^y &= \tfrac{d}{dx} x \\ e^y \tfrac{dy}{dx} &= 1 \\ \tfrac{dy}{dx} &= \frac{1}{e^y} \\ &= \tfrac{1}{x} \end{aligned}$$

Derivative of $y = \ln x$

If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$. If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$.

Derivative of $y = \ln x$

Example

Determine the derivative of $f(x) = 8 \ln x + 5$.

$$\begin{aligned} f'(x) &= 8\tfrac{1}{x} \\ &= \tfrac{8}{x} \end{aligned}$$

Example

Determine the derivative of $f(x) = x^2 \ln x$.

$$\begin{aligned} f'(x) &= 2x \ln x + x^2 \tfrac{1}{x} \\ &= x(2 \ln x + 1) \end{aligned}$$

Derivative of $y = \ln x$

Example

Determine the derivative of $y = 3 \ln^2 5x$.

This one uses the chain rule twice, where $v = 5x$, $u = \ln v$ and $y = 3u^2$.

$$\begin{aligned} \tfrac{dy}{dx} &= \tfrac{dy}{du} \cdot \tfrac{du}{dv} \cdot \tfrac{dv}{dx} \\ &= 6u \cdot \tfrac{1}{v} \cdot 5 \\ &= 30 \ln v \cdot \tfrac{1}{v} \\ &= \frac{30 \ln 5x}{5x} \\ &= \frac{6 \ln 5x}{x} \end{aligned}$$

Derivative of $y = \ln x$

Example

Determine the equation of the tangent to $y = x^2 \ln 3x$ when $x = 2e$.

$$\begin{aligned} \frac{dy}{dx} &= 2x \ln 3x + x^2 \frac{3}{3x} \\ &= 2x \ln 3x + x \end{aligned}$$

When $x = 2e$, $y = 4e^2 \ln 6e$ and $\left.\frac{dy}{dx}\right|_{x=2e} = 4e \ln 6e + 2e$.

Derivative of $y = \ln x$

Substitute $x = 2e$, $y = 4e^2 \ln 6e$ and $m = 4e \ln 6e + 2e$ into $y = mx + b$.

$$\begin{aligned} 4e^2 \ln 6e &= (4e \ln 6e + 2e) \cdot 2e + b \\ &= 8e^2 \ln 6e + 4e^2 + b \\ b &= -4e^2 \ln 6e - 4e^2 \\ &= -4e^2(\ln 6e + 1) \\ &= -4e^2(\ln 6 + 2) \end{aligned}$$

Therefore, the equation of the tangent is $y = (4e \ln 6e + 2e)\, x - 4e^2(\ln 6 + 2)$.

Questions?