MCV4U: Calculus & Vectors

Critical Points and Local Extrema

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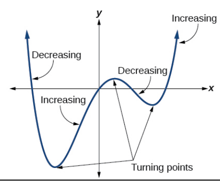


Increasing/Decreasing Intervals

Recap

Determine any increasing/decreasing intervals for the function $f(x) = 2x^3 - 6x^2 - 90x + 7$.

The derivative is $f'(x) = 6x^2 - 12x - 90$, which factors as $f'(x) = 6(x+3)(x-5)$.

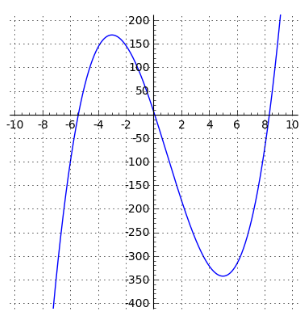
Test values in the intervals $(-\infty, -3)$, $(-3, 5)$ and $(5, \infty)$.

Interval	$(-\infty, -3)$	$(-3, 5)$	$(5, \infty)$
x	-4	0	6
$f'(x)$	54	-90	54
sign	$+$	$-$	$+$

Therefore, $f(x)$ is increasing on $(-\infty, -3)$ and $(5, \infty)$, and decreasing on $(-3, 5)$.

Increasing/Decreasing Intervals

A graph of $f(x)$ confirms these intervals.



Critical Points

In the previous example, the values $x = -3$ and $x = 5$ were special because they were used as interval separators.

Such values are usually called *critical points*. The corresponding output on the function at these points – $f(-3) = x$ and $f(5) = x$, in this case – are called *critical values*.

Critical points indicate special features of a function's graph. These include points at which the function "flattens out", points at which the function changes its direction of opening, discontinuities, and so on.

Critical Points of a Function

For a function $f(x)$, a critical point occurs at $x = c$ when $f(c)$ exists, and either $f'(c) = 0$ or $f'(c)$ is undefined.

Critical Points

Example

Determine any critical values of $y = \frac{3x^2}{x-4}$.

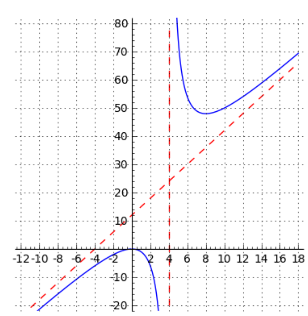
Use the quotient rule to determine the derivative.

$$\begin{aligned}\tfrac{dy}{dx} &= \frac{6x(x-4) - 3x^2(1)}{(x-4)^2} \\ &= \frac{6x^2 - 24x - 3x^2}{(x-4)^2} \\ &= \frac{3x(x-8)}{(x-4)^2}\end{aligned}$$

Since $\frac{dy}{dx} = 0$ when $x = 0$ or $x = 8$, and $\frac{dy}{dx}$ is undefined when $x = 4$, there are three critical values for the function.

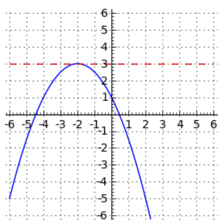
Critical Points

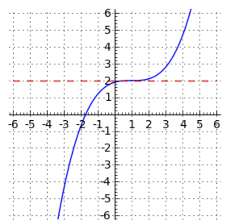
A graph of y shows that y flattens out when $x = 0$ and $x = 8$, and has a vertical asymptote when $x = 4$.



Local Extrema

Recall that if $f'(x) = 0$, a function "flattens out". The function may change from increasing to decreasing (left), *vice versa*, or neither (right).





Local Extrema

A point on a function that is greater than its immediate neighbouring points is called a *local maximum*, while one that is less than its neighbours is called a *local minimum*. Both are examples of *local extrema*.

The left graph on the previous slide had a local maximum at $(-2, 3)$. This point is also the *absolute maximum* – the function never reaches a value higher than 3.

The right graph on the slide had no local minima or maxima. Although the tangent was horizontal at $(1, 2)$, the point was neither higher nor lower than its immediate neighbours.

This illustrates the fact that while all local minima/maxima will occur when $f'(x) = 0$, the converse is not always true – just because $f'(x) = 0$, a function may not have a local extremum at x.

Local Extrema

Example

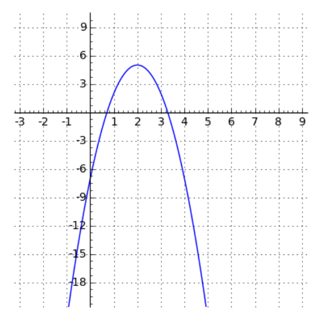
Show that a local maximum occurs for $f(x) = -3x^2 + 12x - 7$, and determine its coordinates.

Since $f(x)$ is a quadratic function, opening downward, it must have a local maximum at its vertex.

Rewriting $f(x)$ in vertex form, $f(x) = -3(x-2)^2 + 5$, with a maximum at $(2, 5)$.

Local Extrema

A graph of $f(x)$ confirms the maximum point.



Local Extrema

Note that the quadratic is increasing on the interval $(-\infty, 2)$ and decreasing on $(2, \infty)$.

This provides us with a simple test for local extrema.

First Derivative Test For Local Extrema

If x is a critical value for $f(x)$, then:

- there is a local maximum at x if $f(x)$ changes from increasing ($f'(x) > 0$) to decreasing ($f'(x) < 0$) at x, or
- there is a local minimum at x if $f(x)$ changes from decreasing ($f'(x) < 0$) to increasing ($f'(x) > 0$) at x.

Note that if $f(x)$ is either increasing on both sides of x, or decreasing on both sides, then x is neither a local maximum nor a local minimum.

Local Extrema

Example

Verify that there is a local maximum for the function $f(x) = -3x^2 + 12x - 7$ on its domain.

The derivative is $f'(x) = -6x + 12$, or $f'(x) = -6(x-2)$.

Thus, there is a critical point at $x = 2$, since $f'(2) = 0$.

Testing in the intervals $(-\infty, 2)$ and $(2, \infty)$ shows that $f(x)$ changes from increasing to decreasing at $x = 2$.

Interval	$(-\infty, 2)$	$(2, \infty)$
x	0	3
$f'(x)$	12	-6
sign	$+$	$-$

Therefore, there is a local maximum when $x = 2$.

Local Extrema

Example

Determine the maximum value of $y = \ln x - x$.

Around $x = 0$, the graph of y should be similar to that for $y = \ln x$, since x is very small.

As x increases in value, $x > \ln x$, so the graph should begin to pull downward, resulting in a local maximum somewhere.

The derivative is $\frac{dy}{dx} = \frac{1}{x} - 1$, or $\frac{dy}{dx} = \frac{1-x}{x}$.

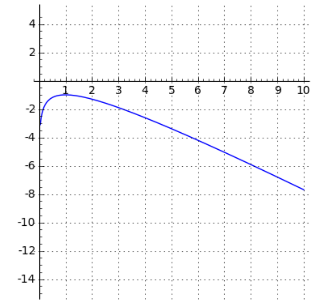
$\frac{dy}{dx}$ is undefined when $x = 0$, due to a vertical asymptote.

Since $\frac{dy}{dx} = 0$ when $x = 1$, a maximum occurs at $\ln(1) - 1 = -1$.

Therefore, the maximum value is -1.

Local Extrema

A graph of y shows the maximum at $(1, -1)$.



Local Extrema

Example

Determine any local extrema for the function $f(x) = \dfrac{2^x}{x}$.

Use the quotient rule to determine the derivative.

$$f'(x) = \frac{2^x \ln 2 \cdot x - 2^x}{x^2} = \frac{2^x(x \ln 2 - 1)}{x^2}$$

Any local minima/maxima will occur when the numerator is equal to zero. 2^x will never equal zero, so we concern ourselves only with $x \ln 2 - 1 = 0$, where $x = \frac{1}{\ln 2} \approx 1.443$.

Local Extrema

Test values on either side of $x = \frac{1}{\ln 2}$ to check if there is a local minimum, local maximum, or neither.

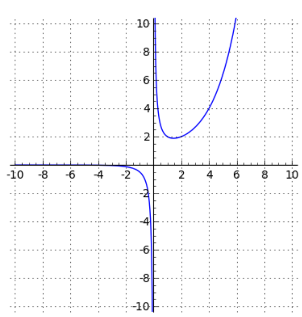
Interval	$\left(0, \frac{1}{\ln 2}\right)$	$\left(\frac{1}{\ln 2}, \infty\right)$
x	1	2
$f'(x)$	-0.614	0.386
sign	$-$	$+$

Therefore, there is a local minimum when $x = \frac{1}{\ln 2}$.

Since $f\left(\frac{1}{\ln 2}\right) = e \ln 2$, this minimum occurs at $\left(\frac{1}{\ln 2}, e \ln 2\right)$, or approximately $(1.443, 1.884)$.

Local Extrema

A graph of $f(x)$ shows the local minimum at $\left(\frac{1}{\ln x}, e \ln 2\right)$.



Questions?