MCV4U: Calculus & Vectors

# Continuity

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## Limits

**Recap**

Determine $\lim_{x\to35} \frac{\sqrt[3]{x-27}-2}{x-35}$ algebraically.

Let $u = \sqrt[3]{x-27}$. Thus, $x = u^3 + 27$ and as $x \to 35$, $u \to 2$.

$$\begin{aligned}
\lim_{x\to35} \frac{\sqrt[3]{x-27}-2}{x-35} &= \lim_{u\to2} \frac{u-2}{u^3-8} \\
&= \lim_{u\to2} \frac{u-2}{(u-2)(u^2+2u+4)} \\
&= \lim_{u\to2} \frac{1}{(u^2+2u+4)} \\
&= \frac{\lim_{u\to2} 1}{\lim_{u\to2} u^2 + 2\lim_{u\to2} u + \lim_{u\to2} 4} \\
&= \tfrac{1}{12}
\end{aligned}$$





## Continuity

Informally, a function is *continuous* if its graph can be drawn without lifting a pencil from the page.

Alternatively, a function can be continuous on one or more intervals as specified, or at a given point.

**Continuity of a Function**

A function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

If the left- and right-handed limits exist, and have the same value as the function itself, then there are no breaks or holes in the graph.

Functions that are not continuous are *discontinuous*.





## Types of Discontinuities

A function is discontinuous if it contains one or more of the following four types of discontinuities.

1. removable (point) discontinuity
2. jump discontinuity
3. infinite discontinuity
4. essential discontinuity





## Types of Discontinuities

A *removable discontinuity* occurs when $\lim_{x\to a} f(x)$ exists and is some finite value, but $\lim_{x\to a} f(x) \neq f(a)$.

A function with a removable discontinuity at $x = a$ will have a "hole" in its graph at $a$.

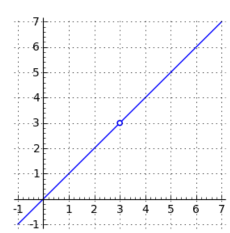
The function may or may not be defined at $f(a)$. If it is defined, there will be a single point at $a$ that is some distance from the rest of the function.

Removable discontinuities typically occur when rational functions have a factor cancelled from the numerator and denominator, or through piecewise functions.

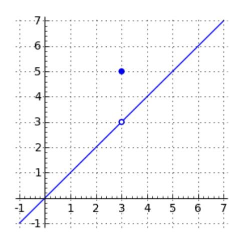




## Types of Discontinuities



A removable discontinuity at $x = 3$ for $f(x) = \frac{x^2-3x}{x-3}$.



A removable discontinuity at $x = 3$ for $f(x) = \begin{cases} x, x \neq 3 \\ 5, x = 3 \end{cases}$.

## Types of Discontinuities

A *jump discontinuity* occurs when the left- and right-handed limits at $x = a$ exist and are finite, but are not equal.

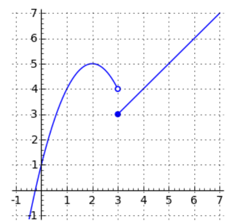
A function with a jump discontinuity will see its graph "jump" up or down to a new value.

Jump discontinuities seldom occur in common functions (e.g. polynomials, exponential, logarithmic), but may be described by piecewise functions or by "square-wave" functions.

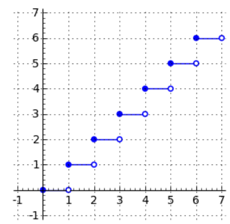




## Types of Discontinuities



A jump discontinuity at $x = 3$ for $f(x) = \begin{cases} -(x-2)^2+5, x \leq 3 \\ 1, x \geq 3 \end{cases}$.



Multiple jump discontinuities for $f(x) = \lfloor x \rfloor$.





## Types of Discontinuities

An *infinite discontinuity* occurs when the left- and right-handed limits exist, and at least one of them is $\pm\infty$.

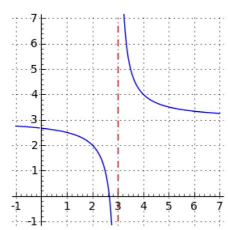
Vertical asymptotes are examples of infinite discontinuities, and occur frequently in rational functions, logarithmic functions, and many others.

Note that the left- and right-handed limits may be different. For instance, a function will have an infinite discontinuity at $x = a$ if $\lim_{x \to a^-} f(x) = \infty$ and $\lim_{x \to a^+} f(x) = -\infty$.

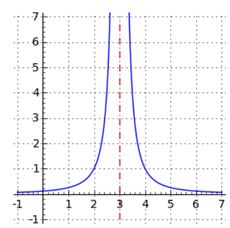




## Types of Discontinuities



An infinite discontinuity at $x = 3$ for $f(x) = \frac{3x-2}{x-3}$.



An infinite discontinuity at $x = 3$ for $f(x) = \frac{1}{(x-3)^2}$.





## Types of Discontinuities

An *essential discontinuity* occurs in all other cases.

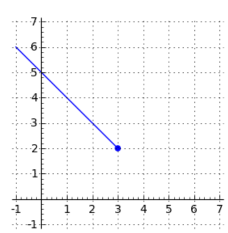
A function may not be defined over some interval, or one of the left- or right-handed limits may not exist.

Essential discontinuities rarely occur for the functions studied in this course; however, the square root function has an essential discontinuity at $x = 0$.

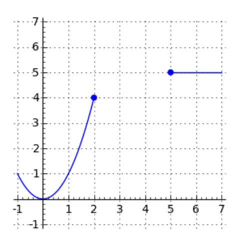




## Types of Discontinuities



An essential discontinuity for $f(x) = -x + 5$, since $\lim_{x \to 3^+} f(x)$ does not exist.



An essential discontinuity at $x = 3$ for $f(x) = \begin{cases} x^2, x \leq 2 \\ 5, x \geq 5 \end{cases}$.

## Testing For Continuity

**Example**

Determine whether the function $f(x) = \begin{cases} x^2, x \leq 2 \\ x+2, x > 2 \end{cases}$ is continuous at $x = 2$ or not, and if $f(x)$ is continuous. If not, describe any discontinuities.

Check the value at $x = 2$ to see if there is a break in the graph.

Since both $2^2 = 4$ and $2 + 2 = 4$, the piecewise function is continuous at $x = 2$.

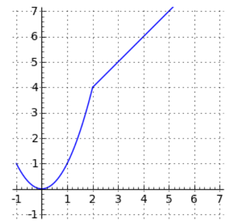
Since both $x^2$ and $x + 2$ are polynomials, $f(x)$ is continuous.





## Testing For Continuity

A graph of $f(x)$ is below.







## Testing For Continuity

**Example**

Determine whether the function $f(x) = \begin{cases} 5, x \leq 3 \\ -x+7, x > 3 \end{cases}$ is continuous at $x = 3$ or not, and if $f(x)$ is continuous. If not, describe any discontinuities.

Check the value at $x = 3$ to see if there is a break in the graph.

Since $-3 + 7 = 4$, but is 5 for all $x \leq 3$, the piecewise function has a jump discontinuity at $x = 3$.

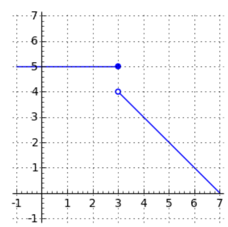
Since there is a discontinuity, $f(x)$ is discontinuous.





## Testing For Continuity

A graph of $f(x)$ is below.







## Testing For Continuity

**Example**

Determine whether the function $f(x) = \tan x$ is continuous at $x = \pi$ or not, and if $f(x)$ is continuous. If not, describe any discontinuities.

Recall that $\tan x$ has vertical asymptotes (infinite discontinuities) at $x = \frac{(2k+1)\pi}{2}, k \in Z$. Therefore, $f(x) = \tan x$ is not continuous.

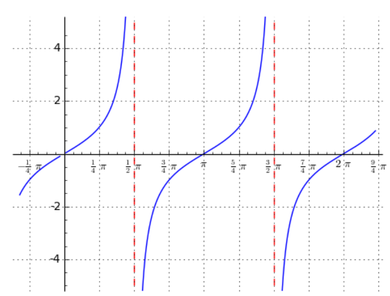
It is, however, continuous at $x = \pi$, since $\lim\limits_{x \to \pi^-} = \lim\limits_{x \to \pi^+} = \tan \pi = 0$.





## Testing For Continuity

A graph of $f(x) = \tan x$ is below.



# Questions?