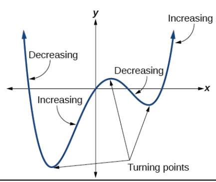
MCV4U: Calculus & Vectors

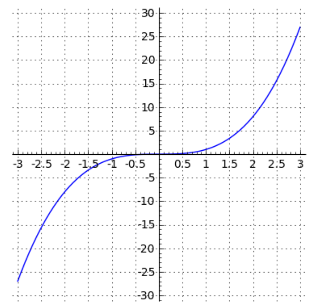
Concavity and Points of Inflection

J. Garvin



Concavity

Consider the graph of $y = x^3$ below.



Concavity

The section of $y = x^3$ on the interval $(-\infty, 0)$ opens downward. This is known as *concave down*.

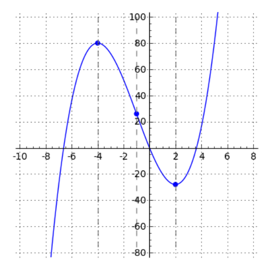
The section on the interval $(0, \infty)$ opens upward. This is known as *concave up*.

Functions are often divided into intervals, depending on their *concavity*.

Concavity

Example

State the intervals on which the function below is concave up and concave down.



Concavity

Moving from left to right, the first interval is $(-\infty, -4)$, which is concave down.

The next interval, $(-4, -1)$, is also concave down. Thus, the two previous intervals are generally grouped as $(-\infty, -1)$ in terms of concavity.

Note that if we were describing intervals of increase or decrease, the critical point $x = -4$ would be a separator instead.

The function is concave up on both $(-1, 2)$ and $(2, \infty)$, or $(-1, \infty)$.

Therefore, the function is concave down on $(-\infty, -1)$ and concave up on $(-1, \infty)$.

Concavity

Since the second derivative represents the rate of change of the slopes of the tangents to the function, it can be used to determine if a function is concave up or concave down.

If the second derivative is positive, then the rate of change is increasing, and the function moves upward.

Similarly, if the second derivative is negative, the rate of change is decreasing, and the function moves downward.

Second Derivative Test For Concavity

For any function $f(x)$, then:

- if $f''(x) > 0$, then $f(x)$ is concave up at x, and
- if $f''(x) < 0$, then $f(x)$ is concave down at x

Concavity

Example

Show that the function $f(x) = \dfrac{x+2}{x^2}$ is concave down on $(-\infty, -6)$ and concave up on $(-6, 0) \cup (0, \infty)$.

Find the first derivative using the quotient rule.

$$\begin{aligned} f'(x) &= \frac{x^2 - 2x(x+2)}{x^4} \\ &= -\frac{x+4}{x^3} \end{aligned}$$

Concavity

Use the quotient rule again to determine the second derivative.

$$\begin{aligned} f''(x) &= \frac{x^3 - 3x^2(x+4)}{x^6} \\ &= -\frac{2(x+6)}{x^4} \end{aligned}$$

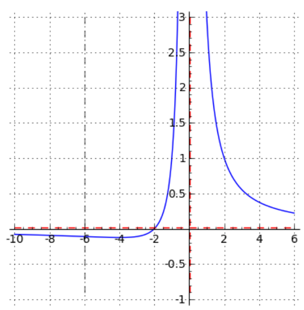
Test values in the intervals $(-\infty, -6)$, $(-6, 0)$ and $(0, \infty)$.

Interval	$(-\infty, -6)$	$(-6, 0)$	$(0, \infty)$
x	-7	-1	1
$f''(x)$	$-\frac{2}{2401}$	10	14
sign	$-$	$+$	$+$

Therefore, $f(x)$ is concave down on $(-\infty, -6)$ and concave up elsewhere.

Concavity

A graph of $f(x)$ suggests that these intervals are valid.



Points of Inflection

What happens if $f''(x) = 0$?

Evaluating the previous example at $x = -6$ gives $f''(-6) = \frac{2(-6+6)}{(-6)^4} = 0$, which is neither concave up nor concave down.

A point at which a function changes concavity (from concave up to concave down, or *vice versa*) is called a *point of inflection*.

Like critical values, points of inflection *may* occur for values of x where $f''(x) = 0$, or $f''(x)$ is undefined.

Since $f(-6) = -\frac{1}{9}$, there is a point of inflection at $\left(-6, -\frac{1}{9}\right)$ in the previous example.

Points of Inflection

Example

Determine any points of inflection for the function $y = x^3 + 3x^2 - 9x - 8$.

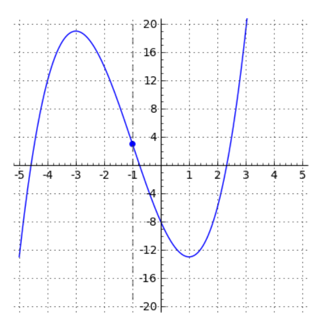
The derivative is $\frac{dy}{dx} = 3x^2 + 6x - 9$, and the second derivative is $\frac{d^2y}{dx^2} = 6x + 6 = 6(x+1)$.

Since $6(x+1) = 0$ when $x = -1$, a point of inflection should occur at this value.

When $x = -1$, $y = (-1)^3 + 3(-1)^2 - 9(-1) - 8 = 3$, so $(-1, 3)$ is a point of inflection.

Points of Inflection

A graph of $y = x^3 + 3x^2 - 9x - 8$ shows how the function changes from concave down to concave up at $(-1, 3)$.



Points of Inflection

The conditions "$f''(x) = 0$" and "$f''(x)$ is undefined" are not sufficient to ensure a point of inflection at x.

Recall that the second derivative represents the rate of change of the slopes of the tangents.

If the slopes change from positive to negative at x, then x might represent a local maximum. Such an interval would be entirely concave down.

Similarly, if the slopes change from negative to positive at x, a local minimum may occur at x and the corresponding interval would be concave up.

Points of Inflection

Example

Determine any points of inflection for the function $f(x) = x^4$.

The derivative is $f'(x) = 4x^3$, and the second derivative is $f''(x) = 12x^2$.

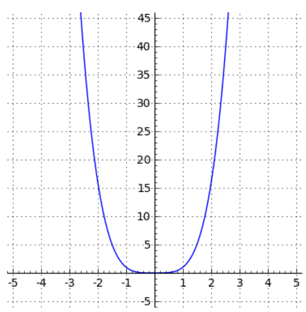
Since $12x^2 = 0$ when $x = 0$, this suggests a point of inflection at $(0, 0)$. Testing on either side of $x = 0$, however, shows that $f(x)$ is concave up everywhere on its domain.

Interval	$(-\infty, 0)$	$(0, \infty)$
x	-1	1
$f''(x)$	12	12
sign	$+$	$+$

Therefore, $(0, 0)$ is a local minimum.

Points of Inflection

Since $f(x) = x^4$ is an even-degree power function with a positive leading coefficient, it is concave up on $(-\infty, \infty)$.



Classifying Extrema

In the previous example, the second derivative was zero at $x = 0$, but there was no inflection point.

Most of the time, the second derivative can tell us if there is a local minimum or local maximum at a given point.

Second Derivative Test For Extrema

If x is a critical point for $f(x)$, then:

- there is a local maximum at x if $f''(x) < 0$, and
- there is a local minimum at x if $f''(x) > 0$.

If $f''(x) = 0$, as in the previous example, then there *may* be a local extremum at x, or there may not be.

Classifying Extrema

Example

Use the second derivative to determine the nature of the critical points of $y = \frac{x^2 + 4}{2x}$.

Determine the first derivative using the quotient rule.

$$\begin{aligned}\tfrac{dy}{dx} &= \frac{2x(2x) - 2(x^2 + 4)}{4x^2} \\ &= \frac{x^2 - 4}{2x^2} \\ &= \frac{(x-2)(x+2)}{2x^2}\end{aligned}$$

Therefore, critical points occur at $x = 0$ and $x = \pm 2$.

Classifying Extrema

When $x = 0$, $\frac{dy}{dx}$ is undefined, so there is a vertical asymptote at $x = 0$.

Find the second derivative using the quotient rule again.

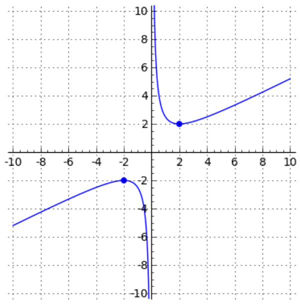
$$\begin{aligned}\tfrac{d^2y}{dx^2} &= \frac{2x(2x^2) - 4x(x^2 - 4)}{4x^4} \\ &= \frac{4}{x^3}\end{aligned}$$

Since $f''(-2) = -\frac{1}{2} < 0$, the graph is concave down on the interval containing $x = -2$, so there is a local maximum when $x = -2$.

Similarly, since $f''(2) = \frac{1}{2} > 0$, the graph is concave up on the interval containing $x = 2$, so there is a local minimum when $x = -2$.

Classifying Extrema

A graph is below. Note that since $\frac{d^2y}{dx^2}$ can never equal 0, there are no points of inflection.



Classifying Extrema

The same conclusion could be reached using the first derivative instead.

Divide the function into four intervals (between critical points and asymptotes) and test values.

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
x	-3	-1	1	3
$f'(x)$	$\frac{5}{18}$	$-\frac{3}{2}$	$-\frac{3}{2}$	$\frac{5}{18}$
sign	$+$	$-$	$-$	$+$

Since $f'(x)$ changes from positive to negative at $x = -2$, the function changes from increasing to decreasing at that point. Therefore, there is a local maximum at $x = -2$.

Similarly, the change from negative to positive at $x = 2$ indicates a local minimum at $x = 2$.

Questions?